# Ergotropy and entanglement in critical spin chains

Begoña Mula,[1,2] Eva M. Fernández,[1] José E. Alvarellos,[1] Julio J. Fernández,[1] David García-Aldea,[1] Silvia N. Santalla,[3] and Javier Rodríguez-Laguna[1,*]

[1]*Departamento de Física Fundamental, Universidad Nacional de Educación a Distancia (UNED), 28040 Madrid, Spain*
[2]*Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain*
[3]*Departamento de Física & GISC, Universidad Carlos III de Madrid, 28911 Leganés, Spain*

(Received 7 August 2022; revised 28 November 2022; accepted 25 January 2023; published 8 February 2023)

A subsystem of an entangled ground state (GS) is in a mixed state. Thus, if we isolate this subsystem from its surroundings, we may be able to extract work applying unitary transformations, up to a maximal amount which is called ergotropy. Once this work has been extracted, the subsystem will still contain some bound energy above its local GS, which can provide valuable information about the entanglement structure. We show that the bound energy for half a free fermionic chain decays as the square of the entanglement entropy divided by the chain length, thus approaching zero for large system sizes, and we conjecture that this relation holds for all one-dimensional critical states.

DOI: 10.1103/PhysRevB.107.075116

## I. INTRODUCTION

Quantum thermodynamics applies the core concepts of quantum information theory [1–3] to design optimal nanoscale devices, such as quantum thermal machines [4–6]. A very fruitful concept is that of *ergotropy* [7,8], i.e., the maximal work that can be reversibly extracted from a mixed state, which is a crucial tool to build efficient quantum batteries [9–11]. Indeed, ergotropy is known to be strongly influenced by the presence of quantum correlations of different types [12–16]. Of course, if we lift the reversibility constraint, we may use quantum measurements to extract work in an optimal way [17,18].

However, the connection works in both directions, and we may employ quantum thermodynamics to characterize the entanglement structure of a quantum system. As is well known, a subsystem of a ground state (GS) is usually not in its local GS. Instead, it must be described by a reduced density matrix, which can be expressed as a thermal density matrix under a certain entanglement Hamiltonian (EH), which need not coincide with the local one [19,20]. Notice that the EH allows us to describe the entanglement structure of complex quantum states in thermal terms. Both the EH and its eigenvalues, which define the entanglement spectrum (ES) [21], have provided invaluable insight into characterizing the entanglement structure of the low-energy states of quantum many-body systems [22–30], in some cases exploiting their conformal invariance [31–33].

In this paper, we introduce the notion of *subsystem ergotropy* within a GS to characterize its entanglement structure through the analysis of the energetic relations between a subsystem $A$ and its environment $B$. The expected value of the local energy of any subsystem will typically exceed its own GS energy, and the subsystem ergotropy is defined as the part that can be extracted in the form of work. Our analysis will focus on a few simple quantum many-body systems, starting with a detailed analysis of free fermionic chains and extending our study to other critical spin chains. In all the considered cases, we benefit from the constraints imposed by conformal invariance on the reduced density matrix. We show that, once the maximal work has been extracted, the remaining bound energy presents universal scaling as the square of the entanglement entropy of the block divided by the system size, thus approaching zero for large system sizes.

This paper is organized as follows. Section II develops the basic theoretical background, combining tools from quantum thermodynamics and quantum information theory. Then we show our analytical and numerical calculations for a free fermionic chain in Sec. III. Other critical spin chains, such as the Ising model in a transverse field or the Heisenberg model, are briefly considered in Sec. IV. This paper ends with a section describing our conclusions and suggestions for further work.

## II. THEORETICAL BACKGROUND

### A. Ergotropy of generic mixed states

The ergotropy $W$ of a mixed state $\rho$ with respect to a Hamiltonian $H$ can be defined as the maximal amount of work that can be extracted from the state by applying unitary operations [7,8], i.e.,

$$W \equiv \max_U [\mathrm{Tr}(\rho H) - \mathrm{Tr}(U \rho U^\dagger H)], \tag{1}$$

where $U$ is any unitary transformation. Alternatively, it can be shown [7] that the ergotropy corresponds to the maximal work that can be reversibly extracted from the system, but the former characterization suits our purposes better. A state defined by a density matrix $\rho$ is called *passive* with respect

*jvrlag@gmail.com

 

to $H$ when its ergotropy is zero, i.e., when we cannot extract any work from it by performing unitary operations. In that case, the eigenstates of $H$ and $\rho$ must be aligned such that the highest probability state of $\rho$ will correspond to the lowest eigenstate of $H$, and so on. Thermal states built on $H$, written as $\rho = Z^{-1}\exp(-\beta H)$ with $\beta = 1/T$ the inverse temperature (we assume $k_B = 1$) and $Z$ the normalization factor, are always passive with respect to their Hamiltonian, but the converse is not true. In other words, all thermal states are passive, but not all passive states are thermal.

Let $p_0 \geqslant p_1 \geqslant \cdots \geqslant p_{N-1}$ be the eigenvalues of $\rho$, where $N$ is the dimension of the Hilbert space. Similarly, let $E_0 \leqslant E_1 \leqslant \cdots \leqslant E_{N-1}$ denote the eigenvalues of $H$ in ascending order, and let $E = \mathrm{Tr}(\rho H)$ be the expected value of the energy of the system. Now let us define the passivized state:

$$\tilde{\rho} \equiv U\rho U^{\dagger}, \tag{2}$$

with $U$ the unitary operator implicitly defined in Eq. (1). Naturally, the spectra of both density matrices must coincide:

$$\mathrm{Sp}(\rho) = \mathrm{Sp}(\tilde{\rho}) = \{p_k\}_{k=0}^{N-1}. \tag{3}$$

Since the passive energy $\widetilde{E} \equiv \mathrm{Tr}(\tilde{\rho}H)$ must be minimal among all density matrices with the same spectrum, we deduce that the maximal probability $p_0$ must share an eigenstate with the GS energy of $H$, $E_0$; the second probability $p_1$ with the first excited state $E_1$, and so on. Therefore,

$$\widetilde{E} = \sum_{k=0}^{N-1} p_k E_k, \tag{4}$$

and degeneracies do not pose any complications. The ergotropy is given by

$$W \equiv E - \widetilde{E} \leqslant E - E_0. \tag{5}$$

Notice that, since we have chosen a common basis of eigenvectors of $H$ and $\tilde{\rho}$, the two operators must commute, $[H, \tilde{\rho}] = 0$. In general, this density matrix $\tilde{\rho}$ need not be thermal for $H$, i.e., it may not be written as $\tilde{\rho} \approx Z^{-1}\exp(-\beta H)$ for any value of $\beta$.

### B. Subsystem ergotropy

Let us consider a quantum system on a composite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with Hamiltonian $H$:

$$H = H_A \otimes I_B + I_A \otimes H_B + H_{AB} \equiv H_0 + H_{AB}, \tag{6}$$

where $H_{\{A,B\}}$ acts on $\mathcal{H}_{\{A,B\}}$, respectively, and $H_{AB}$ will be called the *interaction Hamiltonian*. Of course, this decomposition is not unique, and we will assume that $H_{AB}$ has been chosen as small as possible in some norm. Let $|\Psi\rangle$ be the (nondegenerate) GS energy of $H$, which can always be written as a Schmidt decomposition:

$$|\Psi\rangle = \sum_{k=1}^{\chi} p_k^{1/2} |\phi_k^A\rangle \otimes |\phi_k^B\rangle, \tag{7}$$

where $|\phi_k^A\rangle \in \mathcal{H}_A$ and $|\phi_k^B\rangle \in \mathcal{H}_B$ are two orthornormal sets, $p_k \geqslant 0$ (also in nonincreasing order), and $\chi \leqslant \min[\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)]$ is the Schmidt number. The reduced density matrix for part $A$ can be written as

$$\rho_A = \sum_{k=1}^{\chi} p_k |\phi_k^A\rangle\langle\phi_k^A|. \tag{8}$$

Being positive definite, this matrix can always be written as a thermal density matrix:

$$\rho_A = \exp(-K_A), \tag{9}$$

where $K_A$ is called the EH associated with part $A$. Of course, $K_A$ need not be equal to $H_A$, the local Hamiltonian, and this difference will be crucial in what follows. Also, let us introduce the ES as the spectrum of the EH [21].

Now let us physically separate subsystem $A$ from its environment, i.e., subsystem $B$, by suddenly quenching $H_{AB}$ to zero. The subsequent behavior of our subsystem will be described by $H_A$, with spectrum $\{E_{A,k}\}$, which we may assume to be nondegenerate. We define the three energies involved in our problem:

(1) $E_A = \langle\Psi|H_A \otimes I_B|\Psi\rangle$, the expected value of $H_A$ in the global GS;

(2) $\widetilde{E}_A = \sum_k p_k E_{A,k}$, the passive energy of the system, obtained through unitary transformations; and

(3) $E_{A,0}$, the GS of $H_A$.

These three energies must be in descending order, $E_A \geqslant \widetilde{E}_A \geqslant E_{A,0}$. We define the *excess energy* as $\Delta E_A \equiv E_A - E_{A,0}$. The subsystem ergotropy can be computed as

$$W_A = E_A - \widetilde{E}_A, \tag{10}$$

while

$$Q_A \equiv \widetilde{E}_A - E_{A,0} \tag{11}$$

denotes the amount of energy which is unavailable, which we will call the *subsystem bound energy* [3]. See Fig. 1 for an illustration. The top panel represents the GS of $H$, and $E_A$ is the energy associated with block $A$. The light blue arcs represent the entanglement links [34,35] which characterize the entanglement structure. We reach the middle panel applying a suitable unitary operator on block $A$, maximally reducing its energy to $\widetilde{E}_A$ while preserving the ES and, *a fortiori*, the amount of entanglement with the rest of the system, which

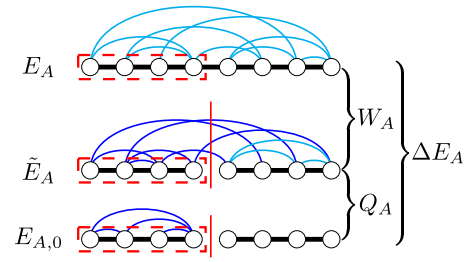


FIG. 1. Illustration of the energies involved in our discussion of the subsystem ergotropy and their differences. Indeed, $E_A$ denotes the expected value of $H_A$ within the global ground state (GS) of $H$, $\widetilde{E}_A$ is the minimal energy achieved through unitary operations on $\mathcal{H}_A$, and $E_{A,0}$ is the GS of $H_A$. Moreover, $\Delta E_A = E_A - E_{A,0}$ is the excess energy, $W_A = E_A - \widetilde{E}_A$ is the subsystem ergotropy, and $Q_A = \widetilde{E}_A - E_{A,0}$ is the subsystem bound energy. The blue arcs denote entanglement, as explained in the text. Notice that, to define these energies, block $A$ must be physically separated from its environment.

in this figure is represented by the number of links leaving $A$. The newly established links are now denoted in dark blue. Finally, the lowest panel denotes the GS of $H_A$, which is now disentangled from the environment, with energy $E_{A,0}$.

### C. Ergotropy and time evolution

Once we have split the subsystem $A$ from its environment, it will evolve under the action of its local Hamiltonian, $H_A$, following von Neumann's equation:

$$i\hbar\,\partial_t \rho_A = [H_A, \rho_A]. \tag{12}$$

Remarkably, this time evolution preserves both the expected value of the energy $E_A$ and the full spectrum of the density matrix, even though the subsequent dynamics can be complex [36–38]. It is relevant to ask how much work we can obtain from this time-evolved density matrix employing unitary transformations, i.e., how the ergotropy evolves after the split quench. The answer is that the ergotropy is exactly preserved along the time evolution. A proof of this fact is straightforward. The time-evolved density matrix for the subsystem after the split can be written as $\rho_A(t) = V(t)\rho_A(0)V^\dagger(t)$ for some unitary transformation $V(t)$. The ergotropy of this matrix, defined in Eq. (1), is the same because the associated passivized state, given in Eq. (2), is the same, if we just use the identity:

$$\tilde{\rho}_A = U\rho_A(0)U^\dagger = UV^\dagger(t)\rho_A(t)V(t)U^\dagger, \tag{13}$$

allowing us to define a new unitary transformation $\tilde{U} = UV^\dagger(t)$, such that $\tilde{\rho}_A = \tilde{U}\rho_A\tilde{U}^\dagger$. This result implies that the work extraction procedure need not start immediately after the disconnection between the subsystem and its environment, if the subsequent evolution is unitary.

### D. Interaction energy inequality

Thus, we can extract work from a subsystem of a composite quantum state in its GS. However, this work should always be less than the corresponding increase in the energy of the system induced by our interaction because, otherwise, the current system energy would be lower than the GS energy $E$. We can prove this result easily. After the unitary transformation on subsystem $A$, the global system will be $|\tilde{\Psi}\rangle$, such that

$$\langle\tilde{\Psi}|H|\tilde{\Psi}\rangle = \widetilde{E} = \widetilde{E}_A + \widetilde{E}_B + \widetilde{E}_{AB}, \tag{14}$$

where each term on the right-hand side corresponds to the expectation value of one of the three operators $H_A$, $H_B$, and $H_{AB}$ on $|\tilde{\Psi}\rangle$, and we notice that $\widetilde{E}_B = E_B$. This energy $\widetilde{E} \geqslant E$, the GS energy, which can be decomposed equally, $E = E_A + E_B + E_{AB}$. Considering that $E_A - \widetilde{E}_A = W_A$, we obtain

$$\widetilde{E}_{AB} - E_{AB} \geqslant W_A \geqslant 0, \tag{15}$$

which implies that the gain through ergotropy must be less or equal than the loss in the interaction term.

## III. ERGOTROPY OF A FREE FERMIONIC CHAIN

We now particularize the previous calculation to the case of a free fermionic chain before extending our results to other critical spin chains. As we will show, the ergotropy and bound energy of free fermionic chains can be explicitly computed and present universal features associated with conformal invariance, like the Casimir energy [39–41]. For simplicity, we will restrict ourselves to the case in which block $A$ corresponds to the left half of the chain.

### A. Free fermionic chains

Let us consider a fermionic chain of $N$ (even) sites with open boundaries, described by the Hamiltonian:

$$H_N = -\sum_{i,j=1}^{N} J_{ij}\, c_i^\dagger c_j, \tag{16}$$

where $c_i^\dagger$ and $c_i$ denote the fermionic creation and annihilation operators on site $i$, and $J_{ij} = \bar{J}_{ji}$ denotes the hopping matrix. We will focus on the homogeneous chain with open boundaries, whose hopping amplitudes are given by $J_{ij} = \delta_{i,j\pm1}$. In this case, the low-energy behavior of the chain can be accurately represented by a conformal field theory (CFT) [42,43].

The GS of the Hamiltonian in Eq. (16) can be obtained through the eigenvalues $\{\varepsilon_k\}$ (in increasing order) and eigenmodes $\{U_{k,i}\}$ of the hopping matrix $J_{ij}$, which are usually called single-body energies and modes, respectively. The spectrum presents particle-hole symmetry $\varepsilon_k = -\varepsilon_{N+1-k}$, and the GS is obtained by filling up the $N/2$ negative energy modes, such that

$$E = \sum_{k=1}^{N/2} \varepsilon_k, \tag{17}$$

while the corresponding eigenstate is a Slater determinant determined by its correlator matrix, defined as

$$C_{ij} \equiv \langle c_i^\dagger c_j\rangle = \sum_{k=1}^{N/2} \bar{U}_{k,i}U_{k,j}. \tag{18}$$

All the entanglement properties can be determined from matrix $C$. Indeed, the reduced density matrix of any block $A$ of size $\ell$ can be obtained diagonalizing the corresponding $\ell\times\ell$ submatrix $C_A$. The set $\{\nu_k^A\}$ of eigenvalues of $C_A$, where each $\nu_k^A \in [0,1]$ determines uniquely the full ES, will be called *entanglement occupations*. The von Neumann entropy of block $A$ can be expressed as [25]

$$S_A = -\sum_{k=1}^{\ell}\left[\nu_k^A \ln\left(\nu_k^A\right) + \left(1-\nu_k^A\right)\ln\left(1-\nu_k^A\right)\right]. \tag{19}$$

Conformal symmetry fixes the universal part of the entanglement entropy of a lateral block $A = \{1,\cdots,\ell\}$ of a critical chain with $N$ sites [44–46]:

$$S_A \approx \frac{c}{6}\ln\left[\frac{N}{\pi}\sin\left(\frac{\pi\ell}{N}\right)\right] + c', \tag{20}$$

where $c = 1$ is the central charge of the associated CFT [42,43], and $c'$ is a nonuniversal constant. Moreover, the EH of a free fermionic chain must also present a free fermionic form, Eq. (16), with a different hopping matrix [19,20]:

$$\rho_A = \frac{1}{Z}\exp(-K_A) = \frac{1}{Z}\exp\left(-\sum_{i,j=1}^{\ell} K_{ij}^A c_i^\dagger c_j\right). \tag{21}$$

The single-body energies of the EH, $\mathcal{E}_k^A$, can be obtained from the entanglement occupations through the Fermi-Dirac expression:

$$\nu_k^A = \frac{1}{1+\exp\left(\mathcal{E}_k^A\right)}, \tag{22}$$

and they are (approximately) equally spaced, with a level separation given by the so-called *entanglement gap* $\mathcal{E}_A \approx \mathcal{E}_{k+1}^A - \mathcal{E}_k^A$, which is known to behave like [32]

$$\mathcal{E}_A \approx \frac{2\pi^2}{\ln(\gamma N)}, \tag{23}$$

where $\ln\gamma \approx 2.3$ is a nonuniversal constant [32]. Moreover, an approximate inverse relation has been proposed between the entanglement gap and the entanglement entropy:

$$\mathcal{E}_A S_A \approx \frac{\pi^2}{3}. \tag{24}$$

### B. Casimir energy and free fermions

Our next aim is to compute the three energies involved in our calculations: $E_A$, $\widetilde{E}_A$, and $E_{A,0}$. Let us start with $E_{A,0}$ for convenience. We proceed to build $H_A$, the hopping matrix for the block $A$, and obtain its eigenvalues $\{\varepsilon_k^A\}_{k=1}^{N/2}$ in increasing order. The GS energy of $A$ is given by

$$E_{A,0} = \sum_{k=1}^{N/4} \varepsilon_k^A. \tag{25}$$

An approximate expression for $E_{A,0}$ as a function of $N$ can be provided [39–41]:

$$E_0(N) = -c_0(N-1) - c_B - \frac{c\pi v_F}{24N} + O(N^{-2}), \tag{26}$$

where we distinguish three terms. The first one, $-c_0(N-1)$, with $c_0 = 2/\pi$, is the *bulk energy*. The second term, $-c_B = -(4/\pi - 1)$, is the *boundary term*. The third one provides the finite-sized correction and is fixed by conformal invariance. Indeed, $c = 1$ is the central charge associated with our theory, and $v_F = 2$ is the Fermi velocity. Thus, we have

$$E_{A,0} \approx -c_0\left(\frac{N}{2} - 1\right) - c_B - \frac{\pi}{6N}. \tag{27}$$

We can use a similar strategy to estimate $E_A$, but we should proceed with care. Indeed, we can obtain $E_A$ numerically from the GS of the whole chain, subtracting the energy associated with the central link and dividing by two:

$$E_A = \frac{E_0(N)}{2} - C_{N/2,N/2+1}. \tag{28}$$

The first term can be easily estimated from Eq. (27):

$$\frac{E_0(N)}{2} \approx -c_0\left(\frac{N}{2} - 1\right) + \frac{c_0}{2} - \frac{c_B}{2} - \frac{\pi}{24N}, \tag{29}$$

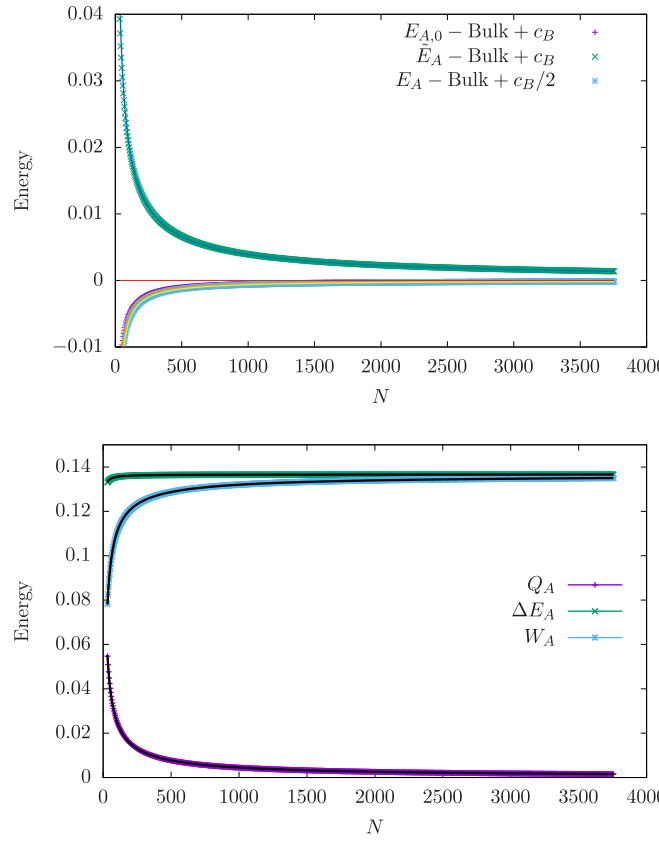


FIG. 2. Top: The three energies involved, $E_A$, $\widetilde{E}$, and $E_{A,0}$, for a free fermionic chain, with $A$ the left half, as a function of the system size, along with the theoretical asymptotic expressions, Eqs. (27), (32), and (39). Bottom: The three energy differences, $\Delta E_A = E_A - E_{A,0}$, $W_A = E_A - \widetilde{E}$, and $Q_A = \widetilde{E} - E_{A,0}$, and their expected theoretical values according to Eqs. (33), (40), and (41)

and the second one can be found by making use of Eq. (18), giving rise to an alternating behavior:

$$C_{n,n+1} \approx -\frac{c_0}{2} - \frac{\pi}{24(N+1)^2} + \frac{(-1)^n}{2(N+1)\sin\left[\frac{\pi(n+1/2)}{N+1}\right]}, \tag{30}$$

which, since $N/2$ is even, reduces for the central link to

$$C_{N/2,N/2+1} \approx -\frac{c_0}{2} - \frac{\pi}{24(N+1)^2} + \frac{1}{2(N+1)}, \tag{31}$$

yielding

$$E_A \approx -c_0\left(\frac{N}{2} - 1\right) - \frac{c_B}{2} - \left(\frac{\pi}{24} + \frac{1}{2}\right)\frac{1}{N}. \tag{32}$$

We notice that the bulk term is the same as for $E_{A,0}$, and the boundary term is exactly half, as we would expect intuitively since this subsystem only possesses one boundary instead of two. We should stress that a naïve calculation would yield a Casimir correction $\pi/(24N)$, but we obtain an additional contribution from the energy associated with the central link. The validity of the approximations to these two energies, $E_{A,0}$ and $E_A$, can be checked in Fig. 2.

Therefore, the excess energy $\Delta E_A = E_A - E_{A,0}$ is given by

$$\Delta E_A \approx \frac{c_B}{2} + \left(\frac{\pi}{8} - \frac{1}{2}\right)\frac{1}{N}. \tag{33}$$

### C. Bound energy and entanglement

Extracting the maximal amount of work through unitary operators reversibly is equivalent to minimizing the block energy while preserving the full spectrum of the reduced density matrix. Thus, we proceed to align the occupation eigenvectors with the eigenstates of $H_A$, whose eigenvalues will be denoted by $\{\varepsilon_k^A\}$. The passive energy $\widetilde{E}_A$ can be written as

$$\widetilde{E}_A = \sum_{k=1}^{\ell} \nu_k \varepsilon_k^A. \tag{34}$$

Since $E_A \leqslant \widetilde{E}_A \leqslant E_{A,0}$, it is reasonable to consider that the passive energy $\widetilde{E}_A$ will also present the same bulk term as in Eq. (27) but with different corrections. Let us provide a similar asymptotic expansion to its value.

The eigenvalues of $H_A$ can be found exactly:

$$\varepsilon_p^A = -2\cos\left(\frac{p\pi}{N/2+1}\right), \tag{35}$$

with $p \in \{1 \cdots N/2\}$, and those of the correlation matrix $C_A$ can also be approximated as

$$\nu_p^A \approx \frac{1}{1+\exp\left[-\beta(p-N/4)\right]}, \tag{36}$$

where $\beta$ corresponds to the entanglement gap, given in Eq. (23) [32]. Thus, the passive energy is given by

$$\widetilde{E}_A = \sum_{p=1}^{N/2} \varepsilon_p^A \nu_p^A \approx \sum_{p=1}^{N/2} \frac{-2\cos(2\pi p/N)}{1+\exp[-\beta(p-N/4)]}. \tag{37}$$

If we take the continuum limit, making use of the Sommerfeld expansion [47] and the Euler-Maclaurin formula, we arrive at

$$\widetilde{E}_A \approx -c_0\left(\frac{N}{2}-1\right) - c_B - \frac{c\pi v_F}{12N} + \frac{2\pi^3}{3N\beta^2}, \tag{38}$$

so we obtain the final form:

$$\widetilde{E}_A \approx -c_0\left(\frac{N}{2}-1\right) - c_B - \frac{\pi}{6N} + \frac{\ln^2(\gamma N)}{6\pi N}. \tag{39}$$

We may now find the analytic expression for the ergotropy:

$$W_A = E_A - \widetilde{E}_A \approx \frac{c_B}{2} + \left(\frac{\pi}{8}-\frac{1}{2}\right)\frac{1}{N} - \frac{\ln^2(\gamma N)}{6\pi N}, \tag{40}$$

where the requirement $W_A \geqslant 0$ demands that $c_B > 0$. This expression can be checked in the bottom panel of Fig. 2. Furthermore, we can estimate the bound energy:

$$Q_A = \widetilde{E}_A - E_{A,0} \approx \frac{\ln^2(\gamma N)}{6\pi N} \geqslant 0, \tag{41}$$

which is unconditionally positive and can also be checked in the bottom panel of Fig. 2. Notice that Eq. (41) implies that the bound energy is directly related to the inverse squared of the entanglement gap of the system or the square of the entanglement entropy. Using Eqs. (20) and (23), we obtain an approximate relation:

$$Q_A N \approx \frac{6}{\pi} S_A^2, \tag{42}$$

which provides a relation between the entanglement entropy of a block of a free fermionic chain and the bound energy associated. Equation (42) is the main prediction of this work, and we conjecture that its validity extends beyond the case of free fermionic chains, to any critical state in one-dimensional (1D) described by a CFT. The validity of this expression can be numerically checked in Fig. 3.

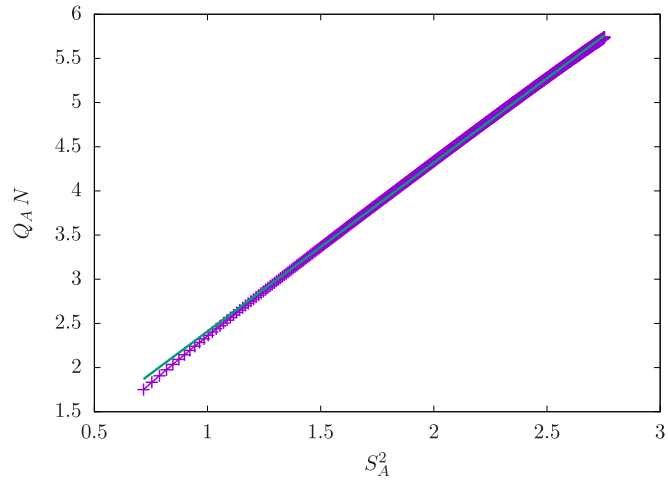


FIG. 3. Numerical check of the linear relation between the bound energy multiplied by the system size $Q_A N$ and the entanglement entropy squared $S_A^2$ for the free fermionic chains, Eq. (42), for sizes $N$ in the same range as in Fig. 2. The slope of the straight line, as expected, is $6/\pi \approx 1.9$.

We may define an ergotropy fraction $w_A = W_A/\Delta E_A$ and a bound fraction $q_A = Q_A/\Delta E_A$ as the ratios between the ergotropy or the bound energy to the excess energy. We can see that $w_A \to 1$ and $q_A \to 0$ as $N \to \infty$, implying that, for larger systems, we can extract most of the excess energy in the form of work using unitary transformations.

## IV. PRELIMINARY RESULTS ON OTHER CRITICAL MODELS

We have considered two other spin chains, the critical Ising model in a transverse field (ITF) and the Heisenberg model, and performed numerical explorations using a combination of Lanczos and exact diagonalization for small systems which provide preliminary numerical evidence of the validity of Eqs. (41) and (42) for these systems.

The Hamiltonian of the ITF model that we have considered is given by

$$H_{\mathrm{ITF}} = -\sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z - \Gamma \sum_{i=1}^{N} \sigma_i^x, \tag{43}$$

for $\Gamma = 1$. The low-energy eigenstates of $H_{\mathrm{ITF}}$ are known to follow a CFT with central charge $c = \frac{1}{2}$ [42,43]. Therefore,

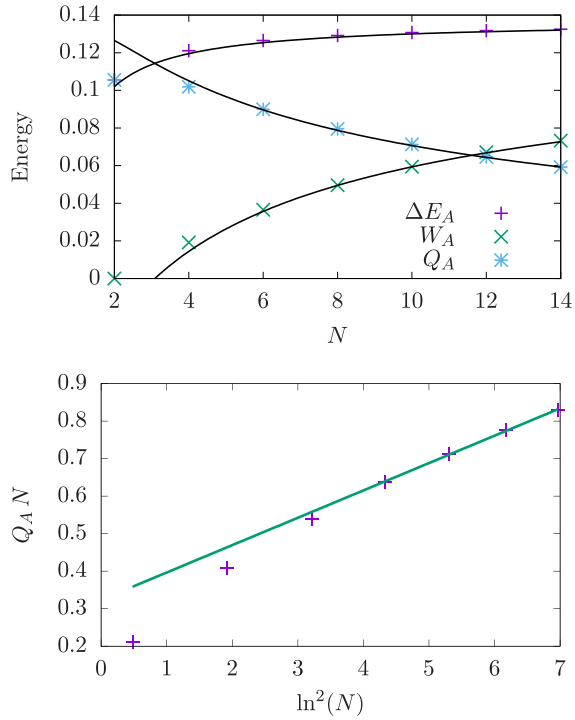


FIG. 4. Subsystem energy decomposition for small Ising critical chains, with $N$ up to 14. Top: Energies $\Delta E_A$, $W_A$, and $Q_A$ for the left half-chain as a function of the system size, along with the expected theoretical fits. Bottom: Approximate linear relation between $Q_A N$ and $\ln^2(N)$, showing the expected relation between $Q_A$ and $S_A$, Eq. (42), along with a linear fit to the last five points.

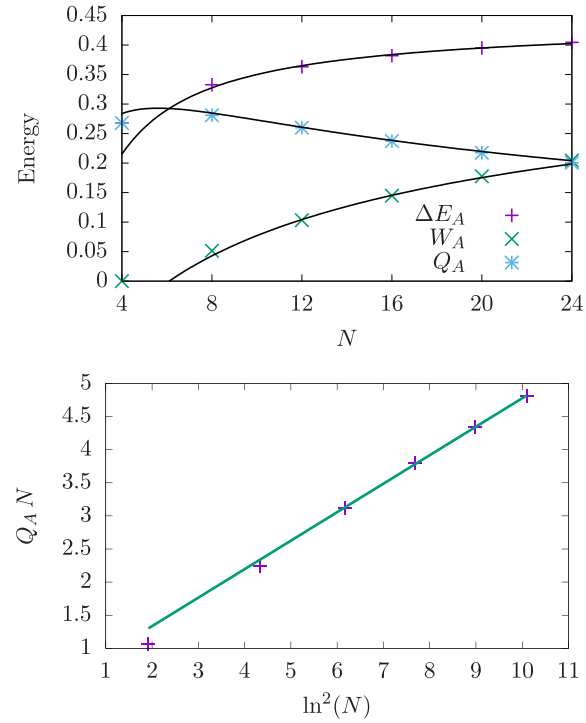


FIG. 5. Subsystem energy decomposition for small Heisenberg chains, with $N$ up to 24, using only multiples of four. Top: Energies $\Delta E_A$, $W_A$, and $Q_A$ for the left half-chain, along with the expected theoretical fits. Bottom: Approximate linear relation between $Q_A N$ and $\ln^2(N)$, showing the expected relation between $Q_A$ and $S_A$, Eq. (42), along with a linear fit to the last five points.

the entanglement entropy of the left half can be written as a linear function of $\ln(N)$. We have obtained preliminary numerical results employing exact diagonalization up to size $N = 14$, which are shown in Fig. 4. In the top panel, we show with points the energy decomposition $\Delta E_A$, $W_A$, and $Q_A$, for the left half-chain of the even-sized systems, along with their fits with continuous lines to theoretical curves suggested by the generalization of Eqs. (33), (40), and (41), i.e.,

$$\begin{aligned} \Delta E_A &\approx \alpha_1 - \frac{\alpha_2}{N}, \\ W_A &\approx \alpha_1 - \frac{\alpha_2}{N} - \alpha_3 \frac{\ln^2(\alpha_4 N)}{N}, \\ Q_A &\approx \alpha_3 \frac{\ln^2(\alpha_4 N)}{N}. \end{aligned} \tag{44}$$

In our case, the optimal values of the parameters are $\alpha_1 \approx 0.137$, $\alpha_2 \approx 0.07$, $\alpha_3 \approx 0.044$, and $\alpha_4 \approx 5.5$. We would like to stress that we fit the 21 points of the three curves using the same values for the $\alpha_i$ parameters. In the bottom panel of Fig. 4, we observe an approximate linear relation between $NQ_A$ and $\ln^2(N)$, as expected, along with a linear fit obtained from the larger systems. Even though the functional form is shown to be approximately correct, we should use these fitting parameters with care, due to the small system size.

On the other hand, we have considered the antiferromagnetic spin-$\frac{1}{2}$ Heisenberg chain with open boundaries, given by the Hamiltonian:

$$H_{\text{Heisenberg}} = \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1}, \tag{45}$$

which also corresponds to a CFT for low energies, with $c = 1$ in this case, and can be mapped to an interacting fermion Hamiltonian using the Jordan-Wigner transformation [42,43]. As mentioned above, the GS can be analytically obtained using the Bethe ansatz, but we have chosen to obtain it using the Lanczos algorithm up to $N = 24$, considering the full SU(2) symmetry of the model. The top panel of Fig. 5 shows the energy decomposition for the left half of the chain, using only values of $N$ which are multiples of four. Again, we plot along a fit of these 18 points to the form in Eq. (44), obtaining approximate parameters $\alpha_1 \approx 0.44$, $\alpha_2 \approx 0.9$, $\alpha_3 \approx 0.41$, and $\alpha_4 \approx 1.32$. The bottom panel of Fig. 5 shows the linear relation between $Q_A N$ and $\ln^2(N)$, highlighting the validity of Eq. (42), again comparing with a linear fit for the largest sizes.

The approximate validity of Eq. (42) in all three models is related to the fact that it only depends on the following:

(1) The Casimir expression for the energy of the GS,

(2) the affine relation between the entanglement entropy and $\ln(N)$, and

(3) the approximate inverse relation between the entanglement entropy and the entanglement gap.

All these relations stem from conformal invariance, a property shared by all three models discussed in this paper.

It would be interesting to check the validity of our preliminary results for larger system sizes in the ITF and Heisenberg cases. The ITF case can be evaluated using a combination of Jordan-Wigner and Bogoliubov transformations. The Heisenberg case is more involved since, e.g., the density matrix renormalization group cannot be used in a straightforward manner [48] because we need to use both the ES and the full energy spectrum of the subsystem.

## V. CONCLUSIONS AND FURTHER WORK

In this paper, we have considered the excess energy possessed by a subsystem of a GS. Part of this excess energy can be extracted via unitary operations, which we call subsystem ergotropy, and part of it cannot be extracted in this way, which we call subsystem bound energy. For concreteness, we have considered 1D systems which present conformal invariance, and we have done the calculations in detail for free fermionic chains, combining numerical calculations with a detailed analysis of the Casimir corrections to the GS energy. The most relevant relation found is a linear functional dependence between the subsystem bound energy and the square of its entanglement entropy divided by the system size. We have shown that this relation is likely to apply to other critical spin chains, thus allowing us to conjecture that its validity will extend to all 1D CFTs.

We would like to stress that, as the system size grows, the fraction of excess energy which can be extracted as work approaches one. In other words, almost all the subsystem energy becomes available in the thermodynamic limit. This result is nontrivial, although it correspondes with our intuition that, for larger systems, we have larger freedom to manipulate the local mixed state. It is relevant to ask how general this result is. For instance, we may wonder about the behavior of the subsystem ergotropy away from criticality, i.e., for dimerized spin chains or for the Ising model with a noncritical value of the transverse field $\Gamma$, or how to extend it to higher-dimensional systems.

Our results encourage further exploration of the application of quantum thermodynamics to the analysis and characterization of entanglement. Beyond the quantitative study of the ergotropy and bound energies, it is relevant to ask about the passive state which we obtain when all the ergotropy has been obtained. Indeed, it must be a thermal state under the EH, but it is also relevant to ask about its properties under its own local Hamiltonian and how these two Hamiltonians relate. Given the relation between the EH and the Unruh effect [49–51], this research program may bear fruits also to the interplay between gravity, entanglement, and thermodynamics.

## ACKNOWLEDGMENTS

We acknowledge very useful discussions with G. Sierra, N. S. Sáenz de Buruaga, K. Zawadzki, S. Singha Roy, J. Hoyos, G. M. Andolina, and the anonymous referees. This paper was funded by the Spanish government through Grants No. PGC2018-094763-B-I00, No. PID2019-105182GB-I00, and No. PID2021-123969NB-I00 and by Comunidad de Madrid (Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M14 and EPUC3M23), in the context of the V Plan Regional de Investigación Científica e Innovación Tecnológica (PRICIT). B.M. acknowledges financial support through Contract No. 2022/167 under the EPUC3M23 line.

[1] J. Gemmer, M. Michel, and G. Mahler, *Quantum Thermodynamics* (Springer, Berlin, Heidelberg, 2004).

[2] *Thermodynamics in the Quantum Regime*, edited by F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer, Cham, 2018).

[3] M. N. Bera, A. Riera, M. Lewenstein, Z. B. Khanian, and A. Winter, Thermodynamics as a consequence of information conservation, Quantum **3**, 121 (2019).

[4] M. Scully, M. S. Zubairy, G. S. Agarwal, and H. Walther, Extracting work from a single heat bath via vanishing quantum coherence, Science **299**, 862 (2003).

[5] A. Ghosh, V. Mukherjee, W. Niedenzu, and G. Kurizki, Are quantum thermodynamic machines better than their classical counterparts? Eur. Phys. J. Spec. Top. **227**, 2043 (2019).

[6] J. J. Fernández, Optimization of energy production in two-qubit heat engines using the ecological function, Quantum Sci. Technol. **7**, 035002 (2022).

[7] A. E. Allahverdyan, R. Balian, and Th. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, Europhys. Lett. **67**, 565 (2004).

[8] R. Alicki and M. Fannes, Entanglement boost for extractable work from ensembles of quantum batteries, Phys. Rev. E **87**, 042123 (2013).

[9] F. Campaioli, F. A. Pollock, F. C. Binder, L. Céleri, J. Goold, S. Vinjanampathy, and K. Modi, Enhancing the Charging Power of Quantum Batteries, Phys. Rev. Lett. **118**, 150601 (2017).

[10] G. M. Andolina, M. Keck, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, Extractable Work, the Role of Correlations, and Asymptotic Freedom in Quantum Batteries, Phys. Rev. Lett. **122**, 047702 (2019).

[11] D. Rosa, D. Rossini, G. M. Andolina, M. Polini, and M. Carrega, Ultra-stable charging of fast-scrambling SYK quantum batteries, J. High Energy Phys. 11 (2020) 067.

[12] G. Francica, J. Goold, F. Plastina, and M. Paternostro, Daemonic ergotropy: Enhanced work extraction from quantum correlations, npj Quantum Inf. **3**, 12 (2017).

[13] G. Francica, F. C. Binder, G. Guarnieri, M. T. Mitchison, J. Goold, and F. Plastina, Quantum Coherence and Ergotropy, Phys. Rev. Lett. **125**, 180603 (2020).

[14] S. Tirone, R. Salvia, and V. Giovannetti, Quantum Energy Lines and the Optimal Output Ergotropy Problem, Phys. Rev. Lett. **127**, 210601 (2021).

[15] A. Touil, B. Çakmak, and S. Deffner, Ergotropy from quantum and classical correlations, J. Phys. A: Math. Theor. **55**, 025301 (2022).

[16] G. Francica, Quantum correlations and ergotropy, Phys. Rev. E **105**, L052101 (2022).
[17] G. Manzano, F. Plastina, and R. Zambrini, Optimal Work Extraction and Thermodynamics of Quantum Measurements and Correlations, Phys. Rev. Lett. **121**, 120602 (2018).
[18] A. Solfanelli, L. Buffoni, A. Cuccoli, and M. Campisi, Maximal energy extraction via quantum measurement, J. Stat. Mech. (2019) 094003.
[19] J. J. Bisognano and E. H. Wichmann, On the duality condition for a Hermitian scalar field, J. Math. Phys. **16**, 985 (1975).
[20] J. J. Bisognano and E. H. Wichmann, On the duality condition for quantum fields, J. Math. Phys. **17**, 303 (1976).
[21] H. Li and F. D. M. Haldane, Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States, Phys. Rev. Lett. **101**, 010504 (2008).
[22] I. Peschel and M.-C. Chung, Density matrices for a chain of oscillators, J. Phys. A: Math. Gen. **32**, 8419 (1999).
[23] M.-C. Chung and I. Peschel, On density-matrix spectra for two-dimensional quantum systems, Phys. Rev. B **62**, 4191 (2000).
[24] M.-C. Chung and I. Peschel, Density-matrix spectra of solvable fermionic systems, Phys. Rev. B **64**, 064412 (2001).
[25] I. Peschel, Calculation of reduced density matrices from correlation functions, J. Phys. A **36**, L205 (2003).
[26] H. Casini and M. Huerta, Reduced density matrix and internal dynamics for multicomponent regions, Class. Quantum Grav. **26**, 185005 (2009).
[27] H. Casini, M. Huerta, and R. Myers, Towards a derivation of holographic entanglement entropy, J. High Energy Phys. 05 (2011) 036.
[28] V. Eisler and I. Peschel, Analytical results for the entanglement Hamiltonian of a free-fermion chain, J. Phys. A: Math. Theor. **50**, 284003 (2017).
[29] H. Santos, J. E. Alvarellos, and J. Rodríguez-Laguna, Entanglement detachment in fermionic systems, Eur. Phys. J. D **72**, 203 (2018).
[30] N. Samos Sáenz de Buruaga, S. N. Santalla, J. Rodríguez-Laguna, and G. Sierra, Piercing the rainbow: Entanglement on an inhomogeneous spin chain with a defect, Phys. Rev. B **101**, 205121 (2020).
[31] J. Cardy and E. Tonni, Entanglement Hamiltonians in two-dimensional conformal field theory, J. Stat. Mech. (2016) 123103.
[32] E. Tonni, J. Rodríguez-Laguna, and G. Sierra, Entanglement Hamiltonian and entanglement contour in inhomogeneous 1D systems, J. Stat. Mech. (2018) 043105.
[33] N. Samos Sáenz de Buruaga, S. N. Santalla, J. Rodríguez-Laguna, and G. Sierra, Entanglement in non-critical inhomogeneous quantum chains, Phys. Rev. B **104**, 195147 (2021).
[34] S. Singha Roy, S. N. Santalla, J. Rodríguez-Laguna, and G. Sierra, Entanglement as geometry and flow, Phys. Rev. B **101**, 195134 (2020).
[35] S. Singha Roy, S. N. Santalla, J. Rodríguez-Laguna, and G. Sierra, Link representation of the entanglement entropies for all bipartitions, J. Phys. A: Math. Theor. **54**, 305301 (2021).
[36] A. Zamora, J. Rodríguez-Laguna, M. Lewenstein, and L. Tagliacozzo, Splitting a critical spin chain, J. Stat. Mech. (2014) P09035.
[37] T. Shimaji, T. Takayanagi, and Z. Wei, Holographic quantum circuits from splitting/joining local quenches, J. High Energy Phys. 03 (2019) 165.
[38] A. Rolph, Local measures of entanglement in black holes and CFTs, SciPost Phys. **12**, 079 (2022).
[39] J. L. Cardy, Conformal invariance and universality in finite-size scaling, J. Phys. A: Math. Gen. **17**, L385 (1984).
[40] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Conformal Invariance, the Central Charge, and the Universal Finite-Size Amplitudes at Criticality, Phys. Rev. Lett. **56**, 742 (1986).
[41] B. Mula, S. N. Santalla, and J. Rodríguez-Laguna, Casimir forces on deformed fermionic chains, Phys. Rev. Res. **3**, 013062 (2021).
[42] P. di Francesco, P. Matthieu, and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997).
[43] G. Mussardo, *Statistical Field Theory* (Oxford University Press, Oxford, 2010).
[44] C. Holzhey, F. Larsen, and F. Wilczek, Geometric and renormalized entropy in conformal field theory, Nucl. Phys. B **424**, 443 (1994).
[45] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Entanglement in Quantum Critical Phenomena, Phys. Rev. Lett. **90**, 227902 (2003).
[46] P. Calabrese and J. Cardy, Entanglement entropy and conformal field theory, J. Phys. A: Math. Theor. **42**, 504005 (2009).
[47] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College Publishing, Philadelphia, 1976).
[48] U. Schollwöck, The density-matrix renormalization group in the age of matrix product states, Ann. Phys. **326**, 96 (2011).
[49] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. D **14**, 870 (1976).
[50] S. Takagi, Vacuum noise and stress induced by uniform accelerator: Hawking-Unruh effect in Rindler manifold of arbitrary dimensions, Prog. Theor. Phys. Suppl. **88**, 1 (1986).
[51] J. Rodríguez-Laguna, L. Tarruell, M. Lewenstein, and A. Celi, Synthetic Unruh effect in cold atoms, Phys. Rev. A **95**, 013627 (2017).