Using Fourier series to obtain cross periodic wall response factors

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Abstract

Wall periodic response factors are a very usual calculation method of transient heat transfer through building envelope elements (walls, roofs…) in steady periodic conditions, used in popular heat load calculation procedures as ASHRAE's RTS method [1]. This response factors, time sampled heat flux responses of a multi-layer wall to a 24h-periodic unit triangle function, can be obtained by means of multiple methods: Laplace's method, state space method, frequency domain methods, etc. These methods are numerical, since there is no analytical way of obtaining these response factors.

The aim of this work is, taking advantage of the periodic nature of excitations, use Fourier series to represent boundary conditions, and this way find an easier and less computationally demanding procedure to calculate these response factors. Additionally, convergence of these Fourier series will be analyzed to determine the minimum set of frequencies needed to ensure a fixed admissible error for wall periodic response factors.

Keywords

Heat transfer in Buildings, Periodic Response Factors, Fourier Series, RTS method

Symbols

A	generic bound.
A_g	(1,1) term of characteristic matrix of the wall.
a_n	n-th cosine Fourier coefficient.
a'_n	n-th cosine Fourier coefficient.
b_n	n-th sine Fourier coefficient. Real form.
b'_n	n-th sine Fourier coefficient. Real form.
B_g	(1,2) term of characteristic matrix of the wall.
C_y	bound for cross transfer function modulus.

c_k	specific heat capacity og layer *k.*
C_g	(2,1) term of characteristic matrix of the wall.
D_g	(2,2) term of characteristic matrix of the wall.
$E(s)$	excitation function in Laplace's space.
f	generic function.
f_n	n-th sine Fourier coefficient. Complex form.
f^m	m-th Fourier approximation of function f.
g	generic function.
$H(s)$	System transfer function.
H_γ	cross transfer function.
i	index.
k	layer index.
K_j	number of layers of j-th massive sheet.
K'	number of non-consecutive massless layers considered in the wall.
ℓ	Laplace transform.
ℓ^{-1}	inverse Laplace transform.
L_k	k-th layer width.
L_{kj}	k-th wythe, j-th layer width.
m	frequency index.
M_g	characteristic matrix of the wall.
M_k	characteristic matrix of layer *k.*
M_f	bound constant for complex form of sine Fourier coefficient.
n	frequency index.
q_e	exterior heat flux function.
q_i	interior heat flux function.
$R(s)$	generic response function in Laplace's space.
$\mathrm{R_k}$	Thermal resistance of layer k
$\mathrm{R_k}$	thermal resistance of massless layer *k.*

R_{i} interior film resistance.

R_{e} exterior film resistance.

s frequency, Laplace transform of variable time.

t time.

T temperature.

T_{e} exterior temperature.

T_{i} interior temperature.

T_{k} temperature of layer *k.*

U overall heat transfer coefficient.

w frequency.

w_{n} *n*-th frequency of Fourier series.

$X_{\Delta}(\mathrm{t})$ exterior heat flux response to an exterior triangular pulse.

$\mathrm{X}_{\Delta}^{\mathrm{k}}$ *k*-th exterior periodic response factor.

$Y_{\Delta}(\mathrm{t})$ interior response heat flux to an exterior triangular pulse.

Y_{Δ}^{k} *k*-th cross periodic response factor.

Y^{m} m-th Fourier approximation of function Y.

Z_{Δ} interior response heat flux to an interior triangular pulse.

Z_{Δ}^{k} k-th interior periodic response factor.

α_{k} thermal diffusivity of layer *k*.

α_{kj} thermal diffusivity of layer *j* wythe *k.*

λ_{k} thermal conductivity of layer *k.*

ρ_{k} density of layer *k.*

Δ triangular pulse function.

$\mathit{\Delta}_{\mathrm{k}}$ triangular pulse function displaced *k* hours.

$\mathit{\Omega}$ characteristic exponent of the wall

$\mathit{\Theta}$ characteristic factor of the wall

$\mathit{\Theta}_{\mathrm{k}}$ characteristic factor of layer *k*

Acronyms

PRF Periodic Response Factors

ASHRAE American Society of Heating, Refrigerating and Air Conditioning Engineers

RTS Radiant Time Series method

RF Response Factor method

PDE Partial Differential Equation

FDR Frequency Domain Response method

1. Introduction.

In the present energy and environmental context, the consideration of buildings as significant energy consumers becomes imperative, owing to their substantial contribution of approximately 33% to the global energy consumption. Of this 33%, about 38% is due to cooling and heating systems [2].

In light of the need to decrease energy usage in buildings, it is essential to prioritize energetically efficient design and efficient management of energy consumption. To achieve this objective, the utilization of building energy simulation tools is crucial.

Within this tools, wall heat conduction through massive building elements (walls and roofs, essentially) is one of the problems to be solved. Owing to the inherent transitory nature of this thermal conduction problem, it necessitates the resolution of a set of coupled partial differential equations (one per each layer of the slab).

When studying systems of the size of a building during a long period of time (a whole year), numerical methods like finite elements or finite differences must be generally discarded because of their computational cost, except for specific issues like thermal bridge effect calculation.

In the middle sixties, Mitalas and Stephenson [3] developed a calculation method called Response Factor Method (RF). This method assumes one-dimensional heat transfer and constant thermal properties of the construction materials, and performs the Laplace transform to the PDEs, solving the problem in Laplace's domain analytically, and going backwards to time domain. This last step is where all the difficulty of the method lies and was solved by Mitalas with the aid of complex residual theory.

This method yields as a result a set of values (**response factors**) which relate the past exterior wall temperatures (or air temperature, depending on if extreme surface radiation-convection problem is included or not) with the heat flux through the extreme lab surfaces.

This set of response factors is made only once for each different slab, and computation of heat flux from them is computationally cheap, which is a clear advantage for long time calculations.

In Mitalas' original method [3], the inversion of the Laplace transform was performed by searching for a series of roots of the denominator of the wall's transfer function. However, this approach led to computationally expensive and delicate iterative methods. Recently, new alternatives have been sought to avoid such computationally expensive iterative procedures [4-15], which shows that this issue continues to generate interest.

A particular case of this problem is when steady periodic external conditions are considered, giving rise to the so-called **periodic response factors (PRF)**. This case is relevant when heat load calculation is considered using ASHRAE Radiant Time Series [16] load calculation method, widely recognized and used.

The aim of this work is to develop a new calculation method for these PRFs, based on the response in linear differential equations to periodic steady state excitations, which has three main advantages over the previously used methods: PRFs can be calculated directly, the algorithm is much simpler, and requires much lower computational cost for the same accuracy than other methods.

2 Methodology

2.1 Definition of the problem

The problem of periodic transient heat conduction through a multi-layer wall (Figure 1) consists in finding out the conduction heat fluxes in the extreme surfaces of the wall $q_e(t)$, $q_i(t)$ knowing the two periodic temperature excitation functions $T_e(t)$, $T_i(t)$ as boundary conditions.

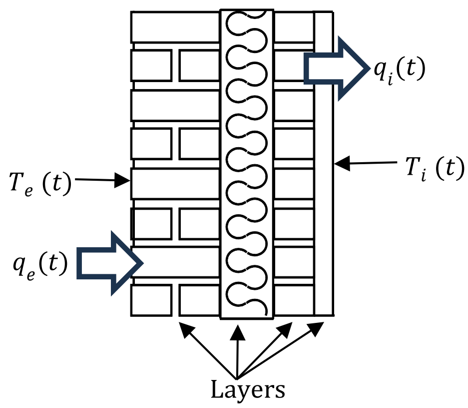

Figure 1. Heat conduction problem in a multi-layer wall.

For the k^{th} layer, heat conduction process is described by the one-dimensional heat equation:

$$\begin{cases} \dfrac{\partial T_k}{\partial t} = \alpha_k \dfrac{\partial^2 T_k}{\partial x^2} \\ T_k(0,t) = T_{ki}(t) \\ T_k(L_k,t) = T_{ke}(t) \\ T(x,0) = T_{k0}(x) \end{cases} \quad (1)$$

where $\alpha_k = \lambda_k / (\rho_k \cdot c_k)$ is the thermal diffusivity of material *k*, λ_k, ρ_k, c_k its thermal conductivity, density and specific heat, respectively, and L_k its thickness. The problem is complete considering that temperatures and heat fluxes in interfaces must coincide:

$$\begin{cases} T_k(0,t) = T_{k-1}(L_{k-1},t) \\ \lambda_k \dfrac{\partial T_k}{\partial x}(0,t) = \lambda_{k-1} \dfrac{\partial T_{k-1}}{\partial x}(L_{k-1},t) \end{cases}, k = 2,\ldots,K \qquad (2)$$

2.2 Laplace's method and periodic Wall Response Factors

To solve the problem (1,2) defined in last section, a common method is the use of Laplace's transform [4,5].

After some transformations, and applying condition (2), the Laplace transforms of the responses of the wall can be written in terms of the Laplace transforms of the excitations in the following way [4,5]:

$$\begin{bmatrix} q_e(s) \\ q_i(s) \end{bmatrix} = \begin{bmatrix} \dfrac{D_g(s)}{B_g(s)} & -\dfrac{1}{B_g(s)} \\ \dfrac{1}{B_g(s)} & -\dfrac{A_g(s)}{B_g(s)} \end{bmatrix} \cdot \begin{bmatrix} T_e(s) \\ T_i(s) \end{bmatrix} \qquad (3)$$

being $M_g(s) = \prod_{k=1}^{K} M_k(s) = \begin{bmatrix} A_g(s) & B_g(s) \\ C_g(s) & D_g(s) \end{bmatrix}$ the **characteristic matrix of the wall**, where

$$M_k(s) = \begin{bmatrix} \cosh\left(L_k \sqrt{\dfrac{s}{\alpha_k}}\right) & \dfrac{\sinh\left(L_k \sqrt{\dfrac{s}{\alpha_k}}\right)}{\lambda_k \sqrt{\dfrac{s}{\alpha_k}}} \\ \lambda_k \sqrt{\dfrac{s}{\alpha_k}} \cosh\left(L_k \sqrt{\dfrac{s}{\alpha_k}}\right) & \cosh\left(L_k \sqrt{\dfrac{s}{\alpha_k}}\right) \end{bmatrix}$$

is the **characteristic matrix of layer *k***, function solely of the thermal properties of the layer.

Thus, the problem is solved in Laplace's space, remaining the issue of inverting the Laplace's transforms of the desired heat fluxes.

To perform this inversion, it is usual to discretize the time in intervals, most of the times hourly, due to availability of excitation temperature data (in climatic records, for example). In this context, our boundary data is a set of hourly periodic time samples of temperature $T_e(k)$, $T_i(k)$, k=1, 2,...,24.

The intermediate values of temperature are estimated by linear interpolation, and this leads to excitation functions written as linear combination of certain basis functions. In the case of usual periodic response factors, unit 24-h periodic triangular functions $\{\Delta_k(t)\}_{k=1}^{24}$ are used, writing then any sampled excitation function as

$$T(t)=\sum_{k=1}^{24} T(k)\cdot\Delta_k(t)$$

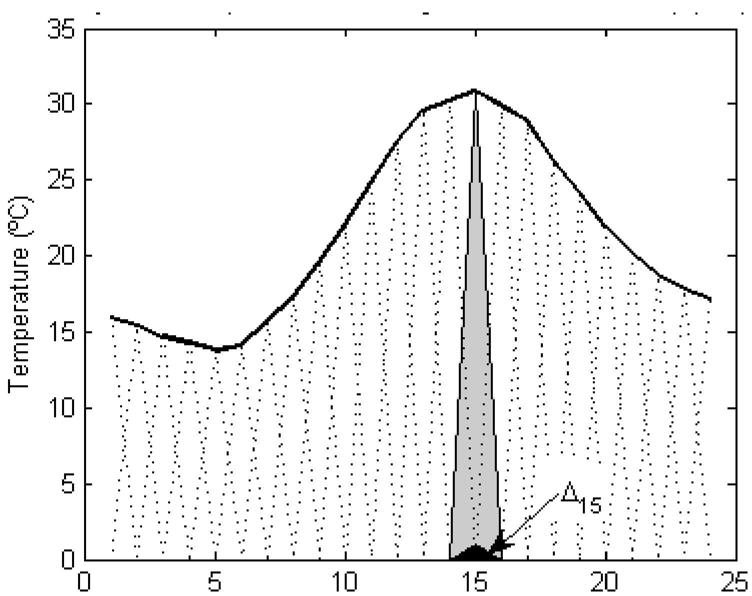

Figure 2. Linear combination of triangular functions. Source: [12]

This basis can be written so that all basis functions $\Delta_k(t)$ are hourly translations of one elementary function $\Delta(t)$, $\Delta_k(t)=\Delta(t-k)$, which we will call hereafter *elemental periodic triangular pulse* (Figure 3):

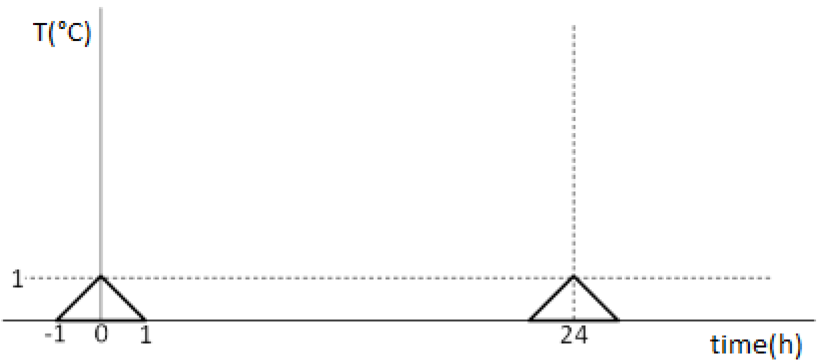

Figure 3. Elemental periodic triangular pulse

Thus, for the boundary conditions, it can be written

$$T(t)=\sum_{k=1}^{24} T(k)\cdot\Delta(t-k).$$

Taking into account the linearity and independence of time of the problem, it is enough to find the flux responses of the wall to the excitation $\Delta(t)$ in the two extreme surfaces of the wall. From equation (3), it is clear that the final response of the wall to any excitation will be then a linear combination of three elementary flux response functions:

- $X_\Delta(t) = q_{\Delta ee}(t)$ exterior heat flux response to an exterior pulse $\Delta(t)$
- $Y_\Delta(t) = q_{\Delta ie}(t)$ interior heat flux response to an exterior pulse $\Delta(t)$
- $Z_\Delta(t) = q_{\Delta ii}(t)$ interior heat flux response to an interior pulse $\Delta(t)$

The general problem has been then reduced to three elemental problems.

Following equation (3), the transforms of the elemental heat fluxes will be

- $X_\Delta(s) = \dfrac{D_g(s)}{B_g(s)}\Delta(s)$
- $Y_\Delta(s) = \dfrac{1}{B_g(s)}\Delta(s)$
- $Z_\Delta(s) = -\dfrac{A_g(s)}{B_g(s)}\Delta(s)$

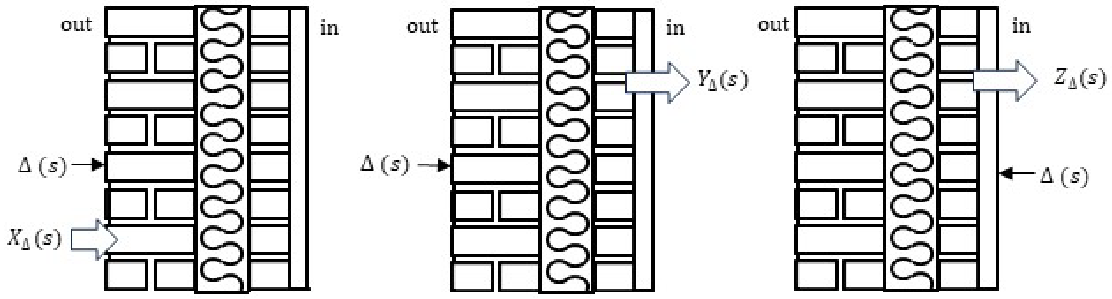

Figure 4. Elementary flux response functions.

Any interior or exterior heat flux $q_i(t)$ or exterior $q_e(t)$ response to any excitation boundary temperature functions $T_e(t)$, $T_i(t)$, the conduction heat fluxes in the extreme surfaces can be written as

$$q_e(t) = \sum_{k=1}^{24} T_e(k) \cdot X_\Delta(t-k) - \sum_{k=1}^{24} T_i(k) \cdot Y_\Delta(t-k)$$

$$q_i(t) = \sum_{k=1}^{24} T_e(k) \cdot Y_\Delta(t-k) + \sum_{k=1}^{24} T_i(k) \cdot Z_\Delta(t-k)$$

The lasting issue is to find the inverse Laplace transforms of the elementary heat fluxes:

$$X_\Delta(t) = \ell^{-1}\left(X_\Delta(s)\right) = \ell^{-1}\left(\frac{D_g(s)}{B_g(s)}\Delta(s)\right)$$

$$Y_\Delta(t) = \ell^{-1}\left(Y_\Delta(s)\right) = \ell^{-1}\left(\frac{1}{B_g(s)}\Delta(s)\right) \quad (4)$$

$$Z_\Delta(t)=\ell^{-1}\left(Z_\Delta(s)\right)=\ell^{-1}\left(-\frac{A_g(s)}{B_g(s)}\Delta(s)\right)$$

The hourly sampled elementary responses $X_\Delta(k)=X_k^\Delta, Y_\Delta(k)=Y_k^\Delta, Z_\Delta(k)=Z_k^\Delta$ are called *periodic wall response factors* and the hourly heat fluxes can finally be written as:

$$q_e(n)=\sum_{k=1}^{24}T_e(n-k)\cdot X_\Delta^k-\sum_{k=1}^{24}T_i(n-k)\cdot Y_\Delta^k$$

$$q_i(n)=\sum_{k=1}^{24}T_e(n-k)\cdot Y_\Delta^k+\sum_{k=1}^{24}T_i(n-k)\cdot Z_\Delta^k$$

ASHRAE's RTS method [1] uses only crossed periodic response factors Y_k^Δ to calculate hourly internal heat flux $q_i(n)$, provided that it assumes constant indoor temperature T_i **:**

$$q_i(n)=\sum_{k=1}^{24}T_e(n-k)\cdot Y_\Delta^k+\sum_{k=1}^{24}T_i\cdot Z_\Delta^k=\sum_{k=1}^{24}T_e(n-k)\cdot Y_\Delta^k-T_i\cdot U$$

where U is the overall heat transmission coefficient of the slab.

2.3. Using Fourier series to calculate periodic response factors.

The complexity of performing the inversion (4) drove the authors in a previous work [12] to avoid the classical method of inversion involving a numerical root finding procedure of the function $B_g(s)$ [17,18], complex and time-consuming in spite of later improvements [5,6], taking advantage of the periodic character of excitations. The developed method also allowed the direct calculation of periodic response factors instead of calculating ordinary response factors and then summing them up periodically [19], or calculated from conduction transfer function coefficients [20,21].

This new method was based on finding a function basis whose functions as excitations made the equation easily invertible. It was found that for the case of periodic continuous excitation functions, a suitable basis was $\left\{\cos\left(\frac{2k\pi t}{T}\right),\sin\left(\frac{2k\pi t}{T}\right)\right\}_{k=0}^{\infty}$, since trigonometric excitations $E(t)=\cos(\omega t)$ in problems of the type

$$R(s)=H(s)E(s)$$

where

- $R(s)$ is the response function,
- $H(s)$ is the system transfer function and
- $E(s)$ is the excitation function.

have a response

$$R(t)=|H(i\omega)|\cdot\cos\left(\omega t+\arg\left(H(i\omega)\right)\right). \quad (5)$$

As the elementary pulse $\Delta(t)$ is continuous and 24h-periodic, it can be written in terms of its Fourier series, since it is a well-known fact [22] that the Fourier Series of any continuous T- periodic function f in an interval $\left[-\frac{T}{2},\frac{T}{2}\right]$ converges uniformly to the original function, this is,

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cdot\cos\left(\frac{2\pi n}{T}\cdot t\right)+\sum_{n=1}^{\infty}b_n\cdot\sin\left(\frac{2\pi n}{T}\cdot t\right)$$

for each $t\in\left[-\frac{T}{2},\frac{T}{2}\right]$, where

$$\begin{cases} a_n=\frac{2}{T}\cdot\int_0^T f(t)\cos\left(\frac{2\pi n}{T}\cdot t\right)dx, n=0,1,... \\ b_n=\frac{2}{T}\cdot\int_0^T f(t)\sin\left(\frac{2\pi n}{T}\cdot t\right)dx, n=1,2,... \end{cases}$$

As $\Delta(t)$ is an even function, $b_n=0$ for every n:

$$\Delta(t)=\frac{1}{24}+\frac{48}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(\pi n/24\right)\cdot\cos\left(\frac{2\pi n}{24}\cdot t\right)$$

$$a_n=\frac{48}{\pi^2}\sin^2\left(\pi n/24\right)\frac{1}{n^2}$$

In figure 5 the successive Fourier approximations of triangular pulse can be seen.

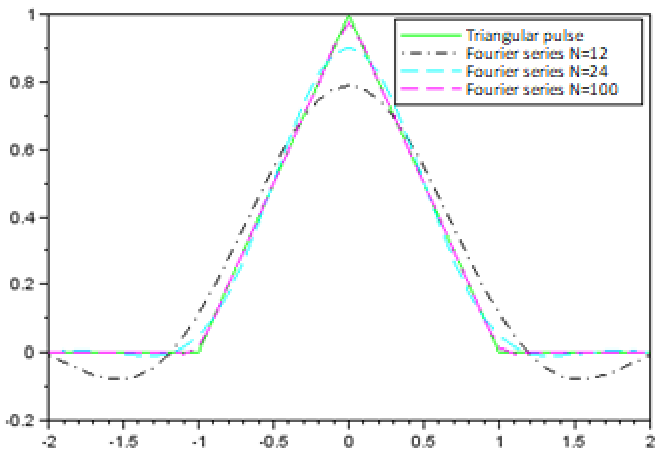

Figure 5: Approximation of a triangle function using Fourier series

Following equation (5), heat flux response can be written as

$$R(t)=\frac{a_0}{2}U+\sum_{n=1}^{\infty}a_n\cdot\left|H\left(i\cdot\frac{2\pi n}{T}\right)\right|\cdot\cos\left(\frac{2\pi n}{T}\cdot t+\arg\left(H\left(i\cdot\frac{2\pi n}{T}\right)\right)\right)$$

Rearranging and using trigonometric properties and definitions,

$$R(t)=\frac{a_0}{2}U+\sum_{n=1}^{\infty}a_n\cdot\mathrm{Re}\left(H\left(i\frac{2\pi n}{T}\right)\right)\cdot\cos\left(\frac{2\pi n}{T}t\right)-\sum_{n=1}^{\infty}a_n\cdot\mathrm{Im}\left(H\left(i\frac{2\pi n}{T}\right)\right)\cdot\sin\left(\frac{2\pi n}{T}t\right)$$

Changing to exponential form:

$$R(t)=\frac{a_0}{2}U+\sum_{n=1}^{\infty}a_n\cdot H\left(i\frac{2\pi n}{T}\right)\cdot\exp\left(i\frac{2\pi n}{T}t\right) \quad (6)$$

Then, the Fourier coefficients of the flux response in the real form of the series are

$$a'_n=a_n\cdot\mathrm{Re}\left(H\left(i\frac{2\pi n}{T}\right)\right)$$

$$b'_n=a_n\cdot\mathrm{Im}\left(H\left(i\frac{2\pi n}{T}\right)\right)$$

and in the complex form

$$f_n=a_n\cdot H\left(i\frac{2\pi n}{T}\right). \qquad (7)$$

Implementation of the method

An example of implementation of this method in the software *scilab* [23] is offered here below:

Main function

```
function Ydelta=facresptriangfourier(layers, m)
  //"layers" is a matrix with layer properties in each row: L,k,rho,cp,R
  //m es the frequency number except w=0
// call function to build characteristic matrix of the wall
exec('C:\Dropbox\scilab\factores de respuesta\caracmat.sci', -1);
 //variable inicialization
 Rglobal=sum(layers(:,5));
 B=[1/Rglobal];
 psi=[0];
 w=%pi/12*[0:m];
  // wall characteristic matrix per frequency
 for k=2:m+1
   Mglobal=eye(2);
  [nlayers,nprop]=size(layers);
  for indexlayer=1:nlayers
    proplayer=layers(indexlayer,:);
    Mglobal=Mglobal*caracmat(%i*w(k)/3600,proplayer);
  end
```

```
  // damping and phase shift storage per frequency
  B=[B abs(1/Mglobal(1,2))];
  psi=[psi atan(imag(1/Mglobal(1,2)),real(1/Mglobal(1,2)))];
 end
 coef_fou=[1/(24)];
 for k=1:m
    coef_fou=[coef_fou 1/12*(sinc(%pi*k/24))^2];
 end
  // construction of periodic cross  response factors
for n=0:23
  Ydelta(n+1)=sum(B.*coef_fou.*cos(w*(n)+psi));
end
endfunction
```

Auxiliary function

```
function M=caracmat(s, properties)
 L=properties(1);
 k=properties(2);
 rho=properties(3);
 cp=properties(4);
 R=properties(5);
 if (cp==0)|(s==0) //massless layers or steady state
  M=[1 R
  0 1];
 else
  alpha=k/(rho*cp);
  factor1=L*sqrt(s/alpha);
  factor2=k*sqrt(s/alpha);
  M=[cosh(factor1) sinh(factor1)/factor2
  sinh(factor1)*factor2 cosh(factor1)];
 end
endfunction
```

As can be seen and was previously mentioned, the code is simple, short, and easy to implement. Additionally, its structure is linear, lacking iterative search processes (like those present in traditional direct root finding method), making it computationally very efficient.

2.4 Truncation error

As we cannot evaluate completely the series (6) and we must truncate to find the desired values, we must estimate the error owing to this truncation:

$$\left|R(t)-R^m(t)\right|=\left|\sum_{n=m+1}^{\infty} a_n \cdot H\left(i\frac{2\pi n}{T}\right)\cdot\exp\left(i\frac{2\pi n}{T}t\right)\right|$$

We find that bounding the Fourier coefficients of the series we can obtain an error convergence order:

For k>1 the **Integral Convergence Criteria** shows that

$$\sum_{n=m+1}^{\infty}\frac{1}{n^k}\leq\int_m^{\infty}\frac{1}{x^k}dx=\frac{1}{k-1}m^{1-k} \qquad (8)$$

Thus, a Fourier coefficient bounding of

$$|f_n|\leq A\frac{1}{n^k}$$

will lead to an error bound for the m^{th} Fourier summation of

$$|f-f^m|\leq\sum_{n=m+1}^{\infty}|f_n|\leq A\sum_{n=m+1}^{\infty}\frac{1}{n^k}\leq\frac{A}{k-1}\frac{1}{m^{k-1}}$$ (k-1th order of convergence).

In order to bound our estimation error, we must then find the behavior of the modulus of Fourier coefficients of the solution

$$|f_n|=|a_n|\cdot\left|H\left(i\frac{2\pi n}{T}\right)\right|$$

when $n\to\infty$.

The behavior of the modulus of the Fourier coefficients $|a_n|$ of the excitation pulse is defined by the smoothness of that excitation function. In general, the smoother the function, the faster its coefficients decrease when $n\to\infty$ [24] and the faster the Fourier series converges to the original function. It can be easily proved that if *f* is *k*-differentiable, and $f^{k+1)}$ is piecewise continuous, then its Fourier coefficients fulfill the condition:

$$|f_n|\leq\frac{M_f}{n^{k+2}} \qquad (9)$$

for a certain constant M_f.

Following (7), once the excitation pulse is chosen, the coefficients a_n are fixed and to define convergence speed of series (6), the behavior of the modulus of the transfer function

$$\left|H\left(i\frac{2\pi n}{T}\right)\right|$$

when $n\to\infty$ must be determined.

This behavior is analyzed in the following section, focusing on the transfer function of cross periodic response factors Y_k^{Δ}, that are the only ones used in the so-called RTS method and that relate the internal heat flow to the outside temperature.

3. Behavior of the multi-layer one-dimensional heat conduction transfer function. Characteristic exponent and factor of the wall and characteristic layer factors.

Let us define the following notation: we will say that the real function $g(x)$ is equivalent to $f(x)$ when $x \to \infty$, and write $g(x) \propto f(x)$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 1$$
.

In a massive layer, the characteristic matrix of the k^{th} wall layer M_k evaluated in the n^{th} frequency $s = i\omega_n$ can be written as

$$M_k(i\omega_n) = \begin{bmatrix} \cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right) & \frac{\sinh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right)}{\lambda_k\sqrt{\frac{i\pi n}{12\alpha_k}}} \\ \lambda_k\sqrt{\frac{i\pi n}{12\alpha_k}}\cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right) & \cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right) \end{bmatrix}$$

while in a massless layer it can be written as

$$M_k(i\omega_n) = \begin{bmatrix} 1 & R_k \\ 0 & 1 \end{bmatrix}.$$

The characteristic matrix of the entire wall will be the product of the characteristic matrices of the wall:

$$M_g(i\omega_n) = \prod_{k=1}^{K} M_k(i\omega_n) = \begin{bmatrix} A_g(i\omega_n) & B_g(i\omega_n) \\ C_g(i\omega_n) & D_g(i\omega_n) \end{bmatrix}.$$

We are interested in the behavior of the moduli of the transfer functions corresponding to cross periodic response factors Y_k^{Δ}:

$$|H_Y(i\omega_n)| = \frac{1}{|B(i\omega_n)|}$$

We can differentiate three cases: walls composed only by massive layers (not including interior and exterior surface resistances for convection-radiation exchange), walls with one or more non-consecutive massless layers (if two massless layers are consecutive, it is treated as one single massless layer with thermal resistance the sum of both resistances), and the case of extra surface resistances.

We will analyze each case separately in the next sections.

3.1 Massive layers and no interior or exterior resistance

It can be proved (see Annex, Theorem 1) that for a wall composed by *K* massive layers and no massless layers:

$$\begin{bmatrix} |A_g(i\omega_n)| & |B_g(i\omega_n)| \\ |C_g(i\omega_n)| & |D_g(i\omega_n)| \end{bmatrix} \propto \frac{1}{2^K} e^{\Omega\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \Theta \cdot \begin{bmatrix} 1 & \frac{1}{\Psi_K\sqrt{\omega_1}}\frac{1}{\sqrt{n}} \\ \Psi_1\sqrt{\omega_1}\sqrt{n} & \frac{\Psi_1}{\Psi_K} \end{bmatrix} \qquad (10)$$

where

- $\omega_1 = \dfrac{2\pi}{T}$,
- $\Omega = \sum_{k=1}^{K} \sqrt{\dfrac{L_k^2}{\alpha_k}}$ is a parameter of the wall which we have called **characteristic exponent of the wall,**
- $\Theta = \prod_{k=1}^{K-1} \left(1 + \dfrac{\Psi_{k+1}}{\Psi_k}\right)$ is a parameter of the wall which we have called **characteristic factor of the wall, and**
- $\Psi_k = \sqrt{\lambda_k \cdot \rho_k \cdot c_k} = \sqrt{\dfrac{\lambda_k^2}{\alpha_k}}$ is a parameter of the layer *k* called the **characteristic factor of layer** *k*.

Thus, Y factors have a transfer function modulus

$$|H_Y(i\omega_n)| = \frac{1}{|B(i\omega_n)|} \propto C_Y \cdot \sqrt{n} \cdot e^{-\Omega\sqrt{\omega_1/2}\sqrt{n}}$$

where $C_Y = 2^K \dfrac{\Psi_K \cdot \sqrt{\omega_1}}{\Theta}$ is a known constant.

3.2 Embedded massless layers and no surface resistances

We define a **massive sheet** of a wall as the sub-wall consisting of the set of massive layers between two consecutive massless layers.

Defined this way, a wall of this type is made of two or more massive sheets separated one another by a massless layer (Figure 6).

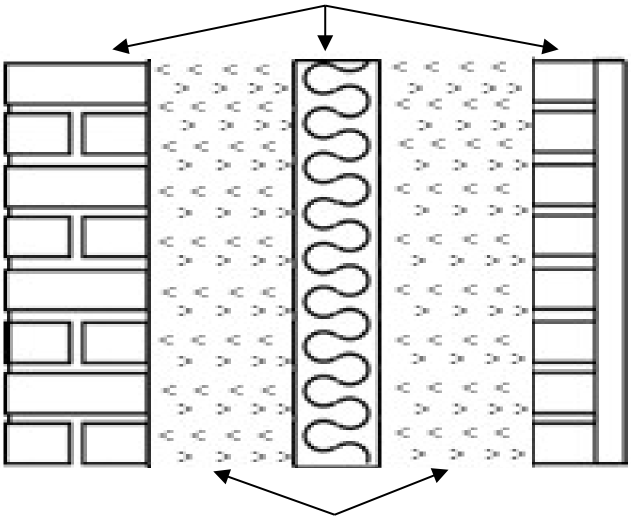

Figure 6. Wall with two massless layers

To obtain a general expression for (10) taking into account embedded massless layers (e.g. air spaces or thin metal layers) it is possible to prove (see Annex, theorem 2) that

$$\left|M_g\left(i\omega_n\right)\right| \propto \frac{1}{2^{\sum_{j=1}^{K'+1} K_j}} e^{\Omega\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \prod_{j=1}^{K'+1} \Theta_j \cdot \prod_{k=1}^{K'} R_k \Psi_{(k+1)_1} \cdot \left(\sqrt{\omega_1}\sqrt{n}\right)^{K'} \begin{bmatrix} 1 & \frac{1}{\Psi_{K'+1,K_{K'+1}}\sqrt{\omega_1}} \frac{1}{\sqrt{n}} \\ \Psi_{11}\sqrt{\omega_1}\sqrt{n} & \frac{\Psi_{11}}{\Psi_{K'+1 K_{K'+1}}} \end{bmatrix}$$

where

- K' is the number of non-consecutive massless layers considered in the wall
- K_j is the number of layers of j^{th} massive sheet, , j=1,…,K'+1.
- Θ_j is the characteristic factor of j^{th} massive sheet j, j=1,…,K'+1.
- R_k is the thermal resistance of massless layer k, k=1,…,K' and
- Ω is obtained only from the massive layers:

$$\Omega = \sum_{j=1}^{K'+1} \sum_{k=1}^{K_j} \sqrt{\frac{L_{kj}^2}{\alpha_{kj}}}$$

Thus, Y factors have a transfer function modulus

$$\left|H_Y\left(i\omega_n\right)\right| = \frac{1}{\left|B\left(i\omega_n\right)\right|} \propto C_Y \cdot \left(\sqrt{n}\right)^{1-K'} \cdot e^{-\Omega\sqrt{\omega_1/2}\sqrt{n}}$$

where $C_Y = 2^{\sum_{j=1}^{K'+1} K_j} \cdot \frac{\Psi_{K'+1,K_{K'+1}} \cdot \left(\sqrt{\omega_1}\right)^{1-K'}}{\prod_{j=1}^{K'+1} \Theta_j \cdot \prod_{k=1}^{K'} R_k \Psi_{(k+1)_1}}$ is a known constant.

3.3 Embedded massless layers and boundary surface resistances

In addition, the expression taking into account inside-outside resistance, which can be modeled as fictitious inside and outside massless layers, is (see Annex, theorem 3):

$$\left|M_g\left(i\omega_n\right)\right| \propto \frac{1}{2^{\sum_{j=1}^{K'-1} K_j}} e^{\Omega\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \prod_{j=1}^{K'+1} \Theta_j \cdot \prod_{k=0}^{K'} R_k \Psi_{(k+1)_1} \cdot \left(\sqrt{\omega_1}\sqrt{n}\right)^{K'+1} \begin{bmatrix} 1 & R_i \\ \frac{1}{R_e} & \frac{R_i}{R_e} \end{bmatrix}$$

with $R_0 = R_e$ the exterior resistance of the wall and

- K' is the number of non-consecutive massless layers considered in the wall.
- K_j is the number of layers of j^{th} massive sheet, , j=1,…,K'+1.
- Θ_j is the characteristic factor of j^{th} massive sheet , j=1,…,K'+1.
- R_k is the thermal resistance of massless layer k, k=1,…,K' and
- Ω is obtained only from the massive layers:

$$\Omega = \sum_{j=1}^{K'+1} \sum_{k=1}^{K_j} \sqrt{\frac{L_{kj}^2}{\alpha_{kj}}}$$

Thus, Y factors have a transfer function modulus

$$\left|H_Y\left(i\omega_n\right)\right| = \frac{1}{\left|B\left(i\omega_n\right)\right|} \propto C_Y \cdot \left(\sqrt{n}\right)^{-1-K'} \cdot e^{-\Omega\sqrt{\omega_1/2}\sqrt{n}}$$

where $C_Y = 2^{\sum_{j=1}^{K'+1} K_j} \cdot \dfrac{\Psi_{K'+1,K_{K'+1}} \cdot \left(\sqrt{\omega_1}\right)^{-1-K'}}{R_i \cdot \prod_{j=1}^{K'+1} \Theta_j \cdot \prod_{k=1}^{K'} R_k \Psi_{(k+1)_1}}$ is a known constant.

4. Error bound and order of convergence.

Let $\Lambda(t)$ be the excitation pulse function, k-differentiable and $\Lambda^{k+1)}(t)$ piecewise continuous, $k \geq 0$.

It has been proved that

$$\frac{\left|H_Y\left(i\omega_n\right)\right|}{\left(\sqrt{n}\right)^{1-K'} \cdot e^{-\Omega\sqrt{\omega_1/2}\sqrt{n}}} \propto C_Y$$

where the definitions of C_Y and Ω depend on the wall type (see sections 3.1,3.2 and 3.3) and K' here accounts for the number of massless layers including boundary surface resistances.

Applying the result (9), it is clear that the Fourier coefficients y_n of the Y heat flux,

$$\left|y_n\right| = \left|\Lambda_n \cdot H_Y\left(i\omega_n\right)\right| \leq \varepsilon_Y \cdot C_Y \frac{1}{n^{k+(K'+3)/2}} \cdot e^{-\Omega\sqrt{\omega_1/2}\sqrt{n}}$$

with ε_Y as near 1 as desired.

As a consequence,

$$\left|Y(t)-Y^m(t)\right|\le C_Y\varepsilon_Y\sum_{n=m+1}^{\infty}e^{-\Omega\sqrt{\omega_1/2}\sqrt{n}}\frac{1}{n^{k+(K'+3)/2}}$$

and using (8) this can be bounded as following:

$$\left|Y(t)-Y^m(t)\right|\le\frac{2C_Y\varepsilon_Y}{2k+(K'+1)}\cdot\frac{1}{m^{k+(K'+1)/2}}e^{-\Omega\sqrt{\omega_1/2}\sqrt{m+1}}$$

Thus, for all wall types, *the convergence for Y factors using Fourier summations is potential-exponential*, with potential order at least $k+(K'+1)/2$, and exponential order at least $\Omega\sqrt{\omega_1/2}$.

In tables 1 and 2 a summary of the error bounds can be found.

Table 1. Convergence and error bound details for multi-layer walls. No surface resistances considered.

Factor type	Error bound	Convergence type	Minimum order	
			Pot.	**Exp.**
Y	$\frac{2C_Y\varepsilon_Y}{2k+(K'+1)}\cdot\frac{1}{m^{k+(K'+1)/2}}e^{-\Omega\sqrt{\omega_1/2}\sqrt{m+1}}$	Potential-exponential	$k+(K'+1)/2$	$\Omega\sqrt{\omega_1/2}$

Table 2. Convergence and error bound details for multi-layer walls. Surface resistances considered.

Factor type	Error bound	Convergence type	Minimum order	
			Pot.	**Exp.**
Y	$\frac{2C_Y\varepsilon_Y}{2k+(K'+1)}\cdot\frac{1}{m^{k+(K'+1)/2}}e^{-\Omega\sqrt{\omega_1/2}\sqrt{m+1}}$	Potential-exponential	$k+(K'+1)/2$	$\Omega\sqrt{\omega_1/2}$

5. Results and discussion

For the verification of the method, a set of 41 walls obtained from the categorization study conducted by Harris and McQuiston [25] has been considered. This study encompasses a wide range of construction solutions that cover almost the entire spectrum of inertias and typical insulation levels found in building enclosures.

The accompanying table 3 presents the results of the number of frequencies (m) required to achieve a precision of 10^{-5} and 10^{-6} in response factors (Y).

Table 3: Minimum required frequencies (m) based on precision for Y response factors .

	m			**m**	
Wall	**10^{-5}**	**10^{-6}**	**Wall**	**10^{-5}**	**10^{-6}**
1	24	47	**22**	11	13
2	21	23	**23**	9	13
3	24	46	**24**	12	16

4	23	25	**25**	10	13
5	20	23	**26**	11	15
6	21	24	**27**	9	12
7	22	24	**28**	9	13
8	21	23	**29**	10	13
9	15	19	**30**	9	12
10	18	21	**31**	8	11
11	17	21	**32**	9	12
12	19	22	**33**	8	11
13	19	22	**34**	9	13
14	16	20	**35**	7	10
15	14	18	**36**	8	11
16	15	19	**37**	7	10
17	12	17	**38**	7	10
18	13	16	**39**	7	10
19	13	18	**40**	7	10
20	15	19	**41**	7	9
21	12	16			

It can be observed that for this type of response factors and these levels of precision, the number of times the characteristic matrix needs to be evaluated ranges from 7 to 47 times, decreasing with the thermal mass of the wall. This number of evaluations is considerably lower than in other methods (RF) and comparable to more recent ones such as FDR (around 50 evaluations [26]).

The method was originally developed to harness the simplicity of solving linear PDEs with periodic excitations. This obviates the need for a numerical method to solve the problem through an infinite superposition (a summation of Fourier harmonics) of simple analytical solutions.

When examining the truncation error of the solution series, the specific nature of this problem (heat conduction equation in multilayer walls) leads to a strong attenuation of high-frequency excitation responses, rendering these harmonics negligible with minimal error. Consequently, the developed method proves to be intriguing due to its remarkable simplicity and low computational overhead, all while maintaining comparable accuracy to other methods for obtaining response factors.

It is essential to note that this method is inherently confined to the acquisition of periodic response factors, constituting a significant limitation and precluding its universal applicability. Nevertheless, we contend that the broader adoption of the RTS method for thermal load calculations (or any other method utilizing periodic exterior temperature excitations) sufficiently justifies its exploration.

5. Conclusions

The developed methodology facilitates the derivation of periodic response factors for multilayer slabs, utilized in the computation of thermal loads in building structures employing the widely adopted RTS method.

This approach obviates the necessity of determining the roots of the denominator in the wall's transfer function, a requisite in the conventional RF method. Instead, it relies on Fourier series and a previously established harmonic method.

The proposed technique is straightforward in its implementation and avoids the use of approximations or iterative root-finding procedures. Additionally, it is analytical (exact), except for the Fourier series truncation within the response analysis.

This method demonstrates rapid convergence for the response factors, with increased speed corresponding to the thermal mass of the wall, resulting in minimal computational expense.

As previously evidenced in [12], this method directly derives periodic response factors without the intermediate step of calculating conventional response factors and subsequent periodic summations. This not only saves computational time but also enhances accuracy.

Another noteworthy advantage is that, by design, the summation of the factors consistently aligns with U (thermal transmittance), regardless of the selected precision. This ensures the preservation of energy conservation in the computation of heat flow.

Data availability statement

No data available.

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Annex

Theorem 1. For a characteristic matrix of a multi-layer massive wall, its behavior in infinity is defined by the following expression:

$$\begin{bmatrix} |A_g(i\omega_n)| & |B_g(i\omega_n)| \\ |C_g(i\omega_n)| & |D_g(i\omega_n)| \end{bmatrix} \propto \frac{1}{2^K} e^{\Omega\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \Theta \cdot \begin{bmatrix} 1 & \frac{1}{\Psi_K \sqrt{\omega_1}} \frac{1}{\sqrt{n}} \\ \Psi_1 \sqrt{\omega_1}\sqrt{n} & \frac{\Psi_1}{\Psi_K} \end{bmatrix}$$

Proof:

To prove the theorem, first recall that

$$\begin{bmatrix} A_g(i\omega_n) & B_g(i\omega_n) \\ C_g(i\omega_n) & D_g(i\omega_n) \end{bmatrix} = \prod_{k=1}^{K} \begin{bmatrix} \cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right) & \frac{\sinh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right)}{\lambda_k\sqrt{\frac{i\pi n}{12\alpha_k}}} \\ \lambda_k\sqrt{\frac{i\pi n}{12\alpha_k}}\cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right) & \cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right) \end{bmatrix}$$

It is immediate to show that

$$\begin{bmatrix} \left|\cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right)\right| & \left|\frac{\sinh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right)}{\lambda_k\sqrt{\frac{i\pi n}{12\alpha_k}}}\right| \\ \left|\lambda_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right|\left|\cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right)\right| & \left|\cosh\left(L_k\sqrt{\frac{i\pi n}{12\alpha_k}}\right)\right| \end{bmatrix} \propto \frac{1}{2}\exp\left(L_k\sqrt{\frac{\pi n}{24\alpha_k}}\right) \cdot \begin{bmatrix} 1 & \frac{1}{\lambda_k\sqrt{\frac{\pi n}{12\alpha_k}}} \\ \lambda_k\sqrt{\frac{\pi n}{12\alpha_k}} & 1 \end{bmatrix}$$

Then, applying the definition of Ω and Ψ_k,

$$\begin{bmatrix} |A_g(i\omega_n)| & |B_g(i\omega_n)| \\ |C_g(i\omega_n)| & |D_g(i\omega_n)| \end{bmatrix} \propto \frac{1}{2} e^{\Omega\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \prod_{k=1}^{K} \begin{bmatrix} 1 & \frac{1}{\Psi_k\sqrt{\omega_1 n}} \\ \Psi_k\sqrt{\omega_1 n} & 1 \end{bmatrix}$$

we can immediately apply lemma 1 considering $a_k = 1, b_k = \frac{1}{\Psi_k\sqrt{\omega_1 n}}, c_k = \Psi_k\sqrt{\omega_1 n}$ and the definition of Θ to reach the desired result.

■

Theorem 2. For a characteristic matrix of a multilayer wall with *K'* non-consecutive massive layers, its behavior in infinity is defined by the following expression:

$$\left|M_g\left(i\omega_n\right)\right| \propto \frac{1}{2} e^{\Omega\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \prod_{j=1}^{K'+1} \Theta_j \cdot \prod_{k=1}^{K'} R_k \Psi_{(k+1)_1} \cdot \left(\sqrt{\omega_1}\sqrt{n}\right)^{K'} \begin{bmatrix} 1 & \dfrac{1}{\Psi_{K'+1,K_{K'+1}}\sqrt{\omega_1}} \dfrac{1}{\sqrt{n}} \\ \Psi_{11}\sqrt{\omega_1}\sqrt{n} & \dfrac{\Psi_{11}}{\Psi_{K'+1K_{K'+1}}} \end{bmatrix}$$

Proof:

A multi-layer wall with K' massless layers can be seen as K'+1 massive walls separated one another by a massless layer. The result of Theorem 1 can be applied to each of the K'+1 massive sheets:

$$\left|M_g^1\left(i\omega_n\right)\right| = \prod_{k=1}^{K_1} \begin{bmatrix} \left|A_k\left(i\omega_n\right)\right| & \left|B_k\left(i\omega_n\right)\right| \\ \left|C_k\left(i\omega_n\right)\right| & \left|D_k\left(i\omega_n\right)\right| \end{bmatrix} \propto \frac{1}{2^{K_1}} e^{\Omega_1\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \prod_{k=1}^{K_1} \begin{bmatrix} 1 & \dfrac{1}{\Psi_{1k}\sqrt{\omega_1 n}} \\ \Psi_{1k}\sqrt{\omega_1 n} & 1 \end{bmatrix}$$

$$\left|M_g^2\left(i\omega_n\right)\right| = \prod_{k=1}^{K_2} \begin{bmatrix} \left|A_k\left(i\omega_n\right)\right| & \left|B_k\left(i\omega_n\right)\right| \\ \left|C_k\left(i\omega_n\right)\right| & \left|D_k\left(i\omega_n\right)\right| \end{bmatrix} \propto \frac{1}{2^{K_2}} e^{\Omega_2\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \prod_{k=1}^{K_2} \begin{bmatrix} 1 & \dfrac{1}{\Psi_{2k}\sqrt{\omega_1 n}} \\ \Psi_{2k}\sqrt{\omega_1 n} & 1 \end{bmatrix}$$

...

$$\left|M_g^{K'+1}\left(i\omega_n\right)\right| = \prod_{k=1}^{K_{K'+1}} \begin{bmatrix} \left|A_k\left(i\omega_n\right)\right| & \left|B_k\left(i\omega_n\right)\right| \\ \left|C_k\left(i\omega_n\right)\right| & \left|D_k\left(i\omega_n\right)\right| \end{bmatrix} \propto \frac{1}{2^{K_{K'+1}}} e^{\Omega_1\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \prod_{k=1}^{K_{K'+1}} \begin{bmatrix} 1 & \dfrac{1}{\Psi_{K_{K'+1}k}\sqrt{\omega_1 n}} \\ \Psi_{K_{K'+1}k}\sqrt{\omega_1 n} & 1 \end{bmatrix}$$

Applying lemma 2 with $b_k = \dfrac{1}{\Psi_k\sqrt{\omega_1}}, c_k = \Psi_k\sqrt{\omega_1}$ to the characteristic matrix of the whole wall, product of the characteristic matrices of massive sheets and massless layers,

$$\left|M_g\left(i\omega_n\right)\right| = \left|M_g^1\left(i\omega_n\right)\right| \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \left|M_g^2\left(i\omega_n\right)\right| \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} \cdot \ \cdots \ \cdot \begin{bmatrix} 1 & R_{K'} \\ 0 & 1 \end{bmatrix} \left|M_g^{K'+1}\left(i\omega_n\right)\right|$$

the result is straightforward.

■

Theorem 3. For a characteristic matrix of a multilayer wall with K' non-consecutive massive layers and extreme surfaces resistances considered, its behavior in infinity is defined by the following expression:

$$\left|M_g\left(i\omega_n\right)\right| \propto \frac{1}{2} e^{\Omega\sqrt{\frac{\omega_1}{2}}\sqrt{n}} \cdot \prod_{j=1}^{K'+1} \Theta_j \cdot \prod_{k=0}^{K'} R_k \Psi_{(k+1)_1} \cdot \left(\sqrt{\omega_1}\sqrt{n}\right)^{K'+1} \begin{bmatrix} 1 & R_i \\ 1/R_e & R_i/R_e \end{bmatrix}$$

Proof:

The result is straightforward from Theorem 2 and lemma 3, renaming $R_0 = R_e$ the exterior surface resistance of the wall.

■

Let us prove the following lemmas in order to facilitate the proof of the previous theorems:

Lemma 1:

Let $M_k = \begin{bmatrix} a_k & b_k \\ c_k \cdot a_k & c_k \cdot b_k \end{bmatrix}$ be a finite set of matrices, $a_k, b_k, c_k \in \square$, $k = 1,2,\ldots K$. Then,

$$\prod_{k=1}^{K} \begin{bmatrix} a_k & b_k \\ c_k a_k & c_k b_k \end{bmatrix} = \prod_{k=1}^{K-1} \left(a_k + c_{k+1} b_k \right) \begin{bmatrix} a_K & b_K \\ c_1 a_K & c_1 b_K \end{bmatrix}$$

Proof: We will proceed by induction technique.

For k=1 is immediate since a product with no factors is unity.

Let us consider it true for n,

$$\prod_{k=1}^{n} \begin{bmatrix} a_k & b_k \\ c_k a_k & c_k b_k \end{bmatrix} = \prod_{k=1}^{n-1} \left(a_k + c_{k+1} b_k \right) \begin{bmatrix} a_K & b_K \\ c_1 a_K & c_1 b_K \end{bmatrix}$$

and prove it for n+1:

$$\begin{aligned}
\prod_{k=1}^{n+1} \begin{bmatrix} a_k & b_k \\ c_k a_k & c_k b_k \end{bmatrix} &= \prod_{k=1}^{n} \begin{bmatrix} a_k & b_k \\ c_k a_k & c_k b_k \end{bmatrix} \cdot \begin{bmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} a_{n+1} & c_{n+1} b_{n+1} \end{bmatrix} = \\
&= \prod_{k=1}^{n-1} \left(a_k + c_{k+1} b_k \right) \begin{bmatrix} a_n & b_n \\ c_1 a_n & c_1 b_n \end{bmatrix} \cdot \begin{bmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} a_{n+1} & c_{n+1} b_{n+1} \end{bmatrix} = \\
&= \prod_{k=1}^{n-1} \left(a_k + c_{k+1} b_k \right) \begin{bmatrix} a_{n+1} \left(a_n + b_n c_{n+1} \right) & b_{n+1} \left(a_n + b_n c_{n+1} \right) \\ c_1 a_{n+1} \left(a_n + b_n c_{n+1} \right) & c_1 b_{n+1} \left(a_n + b_n c_{n+1} \right) \end{bmatrix} = \\
&= \prod_{k=1}^{n} \left(a_k + c_{k+1} b_k \right) \begin{bmatrix} a_{n+1} & b_{n+1} \\ c_1 a_{n+1} & c_1 b_{n+1} \end{bmatrix}
\end{aligned}$$

■

Lemma 2:

Let $\begin{bmatrix} 1 & b_k / \sqrt{n} \\ c_k \sqrt{n} & c_k \cdot b_k \end{bmatrix}$, $b_k, c_k \in \square$, $n \in \square$, k=1,2, and $\begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix}$, $R \in \square$,. Then,

$$\begin{bmatrix} 1 & b_1 / \sqrt{n} \\ c_1 \sqrt{n} & c_1 \cdot b_1 \end{bmatrix} \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b_2 / \sqrt{n} \\ c_2 \sqrt{n} & c_2 \cdot b_2 \end{bmatrix} \propto R c_2 \sqrt{n} \begin{bmatrix} 1 & b_2 / \sqrt{n} \\ c_1 \sqrt{n} & c_1 \cdot b_2 \end{bmatrix}$$

Proof:

$$\begin{bmatrix} 1 & b_1/\sqrt{n} \\ c_1\sqrt{n} & c_1 \cdot b_1 \end{bmatrix}\begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b_2/\sqrt{n} \\ c_2\sqrt{n} & c_2 \cdot b_2 \end{bmatrix} = \begin{bmatrix} 1 & b_1/\sqrt{n} \\ c_1\sqrt{n} & c_1 \cdot b_1 \end{bmatrix} \cdot \begin{bmatrix} \left(1+Rc_2\sqrt{n}\right) & b_2/\sqrt{n}\left(1+Rc_2\sqrt{n}\right) \\ c_2\sqrt{n} & c_2 \cdot b_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \left(1+Rc_2\sqrt{n}\right)+b_1c_2 & b_2/\sqrt{n}\left(\left(1+Rc_2\sqrt{n}\right)+b_1c_2\right) \\ c_1\left(\left(1+Rc_2\sqrt{n}\right)+b_1c_2\right) & c_1 \cdot b_2\left(\left(1+Rc_2\sqrt{n}\right)+b_1c_2\right) \end{bmatrix} =$$

$$= \left(\left(1+Rc_2\sqrt{n}\right)+b_1c_2\right)\begin{bmatrix} 1 & b_2/\sqrt{n} \\ c_1\sqrt{n} & c_1 \cdot b_2 \end{bmatrix} \propto Rc_2\sqrt{n}\begin{bmatrix} 1 & b_2/\sqrt{n} \\ c_1\sqrt{n} & c_1 \cdot b_2 \end{bmatrix}$$

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Lemma 3.

Let $\begin{bmatrix} 1 & b/\sqrt{n} \\ c\sqrt{n} & c \cdot b \end{bmatrix}$, $b,c \in \square, n \in \square$, and $\begin{bmatrix} 1 & R_k \\ 0 & 1 \end{bmatrix}$, $R_k \in \square$, k=1,2. Then,

$$\begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b/\sqrt{n} \\ c\sqrt{n} & c \cdot b \end{bmatrix}\begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} \propto cR_1\sqrt{n}\begin{bmatrix} 1 & R_2 \\ 1/R_1 & R_2/R_1 \end{bmatrix}$$

Proof:

$$\begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b/\sqrt{n} \\ c\sqrt{n} & c \cdot b \end{bmatrix}\begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & R_2 + b/\sqrt{n} \\ c\sqrt{n} & c\sqrt{n}\left(R_2 + b/\sqrt{n}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \left(1+R_1c\sqrt{n}\right) & \left(1+R_1c\sqrt{n}\right)\left(R_2+b/\sqrt{n}\right) \\ c\sqrt{n} & c\sqrt{n}\left(R_2+b/\sqrt{n}\right) \end{bmatrix} \propto cR_1\sqrt{n}\begin{bmatrix} 1 & R_2 \\ 1/R_1 & R_2/R_1 \end{bmatrix}$$

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