AUTOMORPHISMS FOR CONNECTIONS ON LIE ALGEBROIDS

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Abstract. Given a Lie algebroid $A \to M$, it is obtained the relation between covariant derivatives and sprays on A. Moreover, it is introduced the notion of an (infinitesimal) automorphism preserving the covariant derivative and its relation with the corresponding spray.

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1. Introduction

The theory of (affine) connections is a classical topic in differential geometry (see, for instance, [8]). Its importance comes from the fact that connections are an essential tool in the study of geometric structures on manifolds. Indeed, a connection allows to compare the local geometry around different points of a manifold: if one wants to study a geometric object in the neighborhood of a point, it is necessary to transport it along a curve, and this is done using a connection.

Given a connection on a manifold M (or more precisely its covariant derivative), in a natural way can be associated a vector field in the tangent bundle TM, the so-called spray associated to the connection. This spray is interesting because it encodes the information related with the connection: for instance, the integral curves of the spray are just the tangent lifts of the geodesics. On the other hand, since we have an extra structure on the manifold, it is natural to study the transformations which preserve the connection, the well-known infinitesimal affine transformations. The relation between the connection and the corresponding spray can be extended to the infinitesimal affine transformations, which can be characterized in terms of the spray and the Lie bracket.

From the point of view of applications, connections and sprays play an important role in the geometric formulation of Lagrangian mechanics [13]. For instance, in [15] time-dependent mechanics can be geometrically described using time-dependent semi-sprays. On the other hand, in control theory, connections provide a valuable tool for studying controllability of simple mechanical control systems (see [2] and references therein).

A natural extension of the notion of the tangent bundle of a manifold as well as the concept of Lie algebra is that of Lie algebroids [10]. They provide the natural framework to develop the geometry of singular geometric structures: a natural example is a Poisson manifold (M, Π), where exists a (generally singular) foliation, which induces a Lie algebroid structure on the cotangent bundle T^*M. On the other hand, Lie algebroids are interesting from the point of view of mechanics, since they are the adequate framework to study lagrangian and hamiltonian systems, in particular under the presence of symmetries. This idea was introduced by A. Weinstein in [17], and developed in [12], where the notion of prolongation of Lie algebroid is used in order to give a formalism which is parallel to the one for Lagrangian Mechanics on the tangent bundle of a manifold.

Connection theory is very limited when used to study objects which presents a singular behaviour. The reason is that if a structure admits a compatible connection then parallel transport will preserve any invariant of the structure, which will not allow any singular behaviour. An example of this restriction appears when dealing with connections compatible with Poisson structures, which force the rank of the Poisson bivector to be constant. However, very interesting Poisson structures have singularities, such as linear ones.

In order to surpass these problems, it has been introduced in [5] the notion of Lie algebroid connection (for the Poisson case, see [16]), in such a way that if the Lie algebroid is the tangent bundle of a manifold, it is recovered the usual notion of a connection. Although there are differences with the classical case, using connections on Lie algebroids, global properties of Lie algebroids has been studied: holonomy, stability of compact leaves... The relation between Lie algebroid connections and sprays was first obtained in [3] in the study of mechanical control systems on Lie algebroids. Moreover, Lie algebroid sprays have been used in [4] to give a new proof of the existence of symplectic realizations of a Poisson manifold and further developed in [1] in order to present a direct construction of a local Lie groupoid integrating a given Lie algebroid. From the point of view of mechanics, in [14] connections are used to give a coordinate-free characterization of the solutions for the Euler-Lagrange equations on Lie algebroids. Furthermore, connections are useful tools in the side of the constitutive theory of materials. In fact, uniformity and homogeneity are characterized by the existence of some kind of (local) connections covering the body $\mathcal{B}$ (see for instance [6, 7]).

Here, we give an alternative (but similar) proof of the relation between Lie algebroid connections and sprays and extend this relation. More precisely, we introduce the notion of ∇ infinitesimal automorphisms for a Lie algebroid connection ∇, which generalize infinitesimal affine transformations, and characterize them in terms of the associated spray. The paper is organized as follows. In Section 2, we recall the notion of Lie algebroid and linear vector fields on them, describing some characterizations first obtained in [11]. After doing this, the definition of covariant derivative is introduced, describing some important associated notions, such as geodesics. In Section 3, we introduce the notion of a spray for a Lie algebroid and relate them with connections on Lie algebroids showing, as a consequence, that the integral curves of the spray corresponding with a connection are just the geodesics. Finally, in Section 4, we define, for a Lie algebroid connection, ∇ infinitesimal automorphisms (being ∇ the covariant derivative of the connection), and characterize them in terms of the associated spray.

2. Lie algebroids, linear vector fields and covariant derivatives

This section is devoted to introduce some basic definitions such as Lie algebroids and linear vector fields and some useful constructions which are going to be necessary for the results in this paper.

2.1. **Lie algebroids.** As a first step, we are going to introduce the well known notion of Lie algebroid [10].

Definition 2.1. *A **Lie algebroid** over M is a triple $(A \to M, \sharp, [\cdot,\cdot])$, where $\pi : A \to M$ is a vector bundle together with a vector bundle morphism $\sharp : A \to TM$, called the anchor, and a Lie bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(A)$, such that the Leibniz rule holds*

$$[\alpha, f\beta] = f[\alpha,\beta] + \sharp(\alpha)(f)\beta, \qquad \alpha,\beta \in \Gamma(A),\ f \in \mathcal{C}^\infty(M). \tag{2.1}$$

Seeing $\sharp$ as a $\mathcal{C}^\infty(M)$-module morphism from $\Gamma(A)$ to $\mathfrak{X}(M)$, for each section $\alpha \in \Gamma(A)$ we are going to denote $\sharp(\alpha)$ by $\alpha^\sharp$. Thus, we have the following fundamental property:

$$[\alpha,\beta]^\sharp = [\alpha^\sharp, \beta^\sharp], \qquad \alpha,\beta \in \Gamma(A). \tag{2.2}$$

Example 2.2. *i) Let M be an arbitrary manifold. Then, the tangent bundle $A = TM \to M$ is naturally a Lie algebroid, with $\sharp = Id_{TM}$ and the Lie bracket on $\Gamma(A) = \mathfrak{X}(A)$ is just the Lie bracket of vector fields.*

ii) Any finite dimensional Lie algebra $\mathfrak{g}$ can be seen as Lie algebroid over a singleton.

iii) Given a Poisson manifold (M,Π), the cotangent bundle $T^*M \to M$ is endowed with a Lie algebroid structure: the anchor is the bundle map $\Pi^\sharp$:

$T^*M \to TM$ asociated to the bivector Π and the Lie bracket on 1-forms is given by

$$[\alpha, \beta] = \mathcal{L}_{\Pi^\sharp(\alpha)}\beta - \mathcal{L}_{\Pi^\sharp(\beta)}\alpha - d(\Pi(\alpha, \beta)), \qquad \alpha, \beta \in \Omega^1(M).$$

Let α be a section of a Lie algebroid. If $U \subset M$ is an open subset such that $\pi^{-1}(U) \cong U \times F$, where F is the fiber of A, α can be written locally as a differentiable map

$$\alpha_U : U \to F.$$

Furthermore, $\sharp$ will be locally expressed as

$$\begin{array}{rccc} \sharp_U : & U \times F & \to & U \times \mathbb{R}^n \\ & (p, v) & \mapsto & (p, \bar{\sharp}_U(p, v)) \end{array}$$

Given a Lie algebroid structure $(\sharp, [\cdot, \cdot])$ on $\pi : A \to M$, a vector bundle isomorphism $\Phi : A \to A$ over $\phi : M \to M$ is said to be a **Lie algebroid isomorphism** if

$$\sharp \circ \Phi = T\phi \circ \sharp, \tag{2.3}$$

$$\Phi_*[\alpha, \beta] = [\Phi_*\alpha, \Phi_*\beta], \qquad \alpha, \beta \in \Gamma(A), \tag{2.4}$$

where $\Phi_*\alpha = \Phi^{-1} \circ \alpha \circ \phi$.

Remark 2.3. *Note that if $f\colon M \to M$ is a diffeomorphism and $x \in \mathfrak{X}(M) \cong \Gamma(TM)$ then $(Tf)_*(x) = Tf^{-1} \circ x \circ f$ is the so-called push-forward of the vector field x by f^{-1} (see, for instance,* [8]*).*

To finish this section, we introduce the suitable notion of path in the Lie algebroid setting.

Definition 2.4. *Let $a \in \mathcal{C}^\infty(I, A)$ be a curve. It is said to be an $A-$**path** if it satisfies*

$$\sharp(a(t)) = \frac{d}{dt}\gamma, \tag{2.5}$$

where $\gamma = \pi \circ a$ is the base path. $P(A)$ will denote the space of A-paths.

2.2. **Linear vector fields and derivations.** Next, we are going to introduce some preliminaries concerning linear vector fields on vector bundles, whose details may be found in [11]. In what follows we will consider a fixed vector bundle $\pi : A \to M$.

Definition 2.5. *A **linear vector field** on A is a pair (ξ, x), where ξ is a vector field on A and x is a vector field on M, such that*

$$\begin{array}{ccc} A & \xrightarrow{\xi} & TA \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle T\pi} \\ M & \xrightarrow[x]{} & TM \end{array}$$

is a morphism of vector bundles. The vector space of linear vector fields on A is denoted by $\mathfrak{X}_{LIN}(A)$.

Observe that this condition implies that for all $\lambda \in \mathbb{R}$,

$$\xi(\lambda \cdot v_p) = T_p\lambda(\xi(v_p)),$$

where $\lambda : A \to A$ is the map given by $\lambda(v_p) = \lambda v_p$. Locally, ξ can be written in the following way,

$$\begin{array}{rccl} \xi_U : & U \times F & \to & \mathbb{R}^n \times F \\ & (p, v) & \mapsto & (x_U(p), \xi_U^1(p, v)), \end{array}$$

where, fixing p, the map $\xi_U^1(p, \cdot) : F \to F$ is a linear map.

Proposition 2.6. [11] *Let ξ be a vector field on A and x be a vector field on M. The following claims are equivalent:*

i) *(ξ, x) is a linear vector field on A.*
ii) *$\xi : \mathcal{C}^\infty(A) \to \mathcal{C}^\infty(A)$ sends fiber-wise linear functions into fiber-wise linear functions and sends basic functions into basic functions.*
iii) *ξ has flow $\{\Phi_t\}$ which are (local) vector bundle morphism over a flow $\{\phi_t\}$ on M.*

A first result that can be proved is the following one.

Proposition 2.7. *Let (ξ, x) and (ν, y) be linear vector field. Then, $([\xi, \nu], [x, y])$ is a linear vector field.*

Given a linear vector field (ξ^*, x) on A^*, condition ii) above shows that ξ^* induces a map

$$\begin{array}{rccl} D_{\xi^*} : & \Gamma(A) & \to & \Gamma(A) \\ & \alpha & \mapsto & D_{\xi^*}\alpha \end{array}$$

characterized by,

$$L_{D_{\xi^*}\alpha} = \xi^*(L_\alpha), \tag{2.6}$$

where we are considering the correspondence between sections of A and fiber-wise linear functions of A^*, $L : \Gamma(A) \to \mathcal{C}^\infty_{LIN}(A^*)$, $\alpha \mapsto L_\alpha$,

$$L_\alpha(f_p) = f_p(\alpha(p)), \qquad f_p \in A_p^*.$$

It is easy to check that for all $f \in \mathcal{C}^\infty(M)$ and $\alpha \in \Gamma(A)$,

$$D_{\xi^*}(f\alpha) = fD_{\xi^*}\alpha + x(f)\alpha.$$

Definition 2.8. *A **derivation** on A is a $\mathbb{R}-$linear map $D : \Gamma(A) \to \Gamma(A)$ with $x \in \mathfrak{X}(M)$ such that,*

$$D(f\alpha) = fD(\alpha) + x(f)\alpha, \qquad \alpha \in \Gamma(A),\ f \in \mathcal{C}^\infty(M).$$

The vector space of these objects is denoted by $Der(A)$.

The map $\xi^* \to D_{\xi^*}$ is an isomorhism of vector spaces. On the other hand, there is an isomorphism $Der(A) \to Der(A^*)$, $D \mapsto D^*$, given by

$$D^*(X)(\alpha) = x(X(\alpha)) - X(D(\alpha)), \qquad X \in \Gamma(A^*),\ \alpha \in \Gamma(A). \tag{2.7}$$

Let (ξ^*, x) be a linear vector field on A^* and D_{ξ^*} its associated derivation on A. If $D^*_{\xi^*}$ is the derivation on A^* given by Eq. (2.7) then (ξ, x) will denote the linear vector field on A such that

$$D_\xi = D^*_{\xi^*}. \tag{2.8}$$

Remark 2.9. If $\{(\Phi_t, \phi_t)\}$ denotes the flow of (ξ, x) and If $\{(\Psi_t, \phi_t)\}$ denotes the flow of (ξ^*, x) then $\Psi_t = \Phi^T_{-t}$, where Φ^T_s is the transpose of Φ_s for each s, i.e.,

$$g_{\phi_t(p)}(\Phi_t(a_p)) = \{\Psi_{-t}(g_{\phi_t(p)})\}(a_p). \tag{2.9}$$

for all $g_{\phi_t(p)} \in A^*_{\phi_t(p)}$ and $a_p \in A_p$.

Proposition 2.10. *For all $\alpha \in \Gamma(A)$,*

$$\frac{d}{dt}_{|t=0}\Big(\Phi_{t*}\alpha\Big) = D_{\xi^*}\alpha.$$

Proof. Let $f_p \in A^*_p$. Using Eq. (2.6),

$$\begin{aligned} f_p(D_{\xi^*}\alpha(p)) &= \xi^*(f_p)(L_\alpha) = \frac{d}{dt}_{|t=0}\Big(L_\alpha \circ \Psi_t(f_p)\Big) = \\ &= \frac{d}{dt}_{|t=0}\Big(\Psi_t(f_p)(\alpha(\phi_t(p)))\Big) = \frac{d}{dt}_{|t=0}\Big(f_p((\Phi_{t*}\alpha)(p))\Big) \\ &= f_p\Big(\frac{d}{dt}_{|t=0}(\Phi_{t*}\alpha)(p)\Big) \end{aligned}$$

□

As a consequence of Proposition 2.10, we have two results that will be useful in the sequel.

Corollary 2.11. *For all s,*

$$\frac{d}{dt}_{|t=s}(\Phi_{t*}\beta) = \Phi_{s*}D_{\xi^*}\beta,$$

$$\Phi_{s*}D_{\xi^*}\beta = D_{\xi^*}\Phi_{s*}\beta.$$

Now, taking into account that for all $X \in \Gamma(A^*)$ and $\alpha \in \Gamma(A)$, $X(\alpha) = L_\alpha \circ X$, and, from Eq. (2.6) and Eq. (2.8),

$$D_\xi(X)(\alpha) = x(X(\alpha)) - \xi^*(L_\alpha) \circ X. \tag{2.10}$$

This identity can be written locally as

$$(D_\xi X)_U(p) = d(X_U)_{|p}(x_U(p)) - {\xi^*}^1_U(p, X_U(p)). \tag{2.11}$$

Now, using this identity for D_{ξ^*} and Eq. (2.6),

$$f(\xi^1_U(p,v)) = -\{{\xi^*}^1_U(p,f)\}(v), \qquad p \in U,\ v \in F,\ f \in F^*. \tag{2.12}$$

Example 2.12. Given a Lie algebroid on $A \to M$ and $\alpha \in \Gamma(A)$, the Leibniz rule (2.1) implies that $[\alpha, \cdot] \in Der(A)$. In this case, the corresponding linear vector field on A, denoted by $(\alpha^c, \alpha^\sharp)$ is the so-called **complete lift** of α and its flow is **(local) flow of the section** α, which is, indeed, a (local) Lie algebroid isomorphism. Moreover, it can be shown that the linear vector field on A^* is the Hamiltonian vector field of the function L_α associated with the linear Poisson structure on A^*, which will be denoted by $\mathcal{H}_{L_\alpha}$.

Example 2.13. If V is a vector space then, any linear automorphism $D\colon V \to V$ is a derivation on V. Note that in this case the linear vector fields are just vector fields whose flow is made of linear automorphisms.

Example 2.14. Let M be a manifold and $T^*M \to M$ be the cotangent bundle of M. If $x \in \mathfrak{X}(M)$ then the Lie derivative $\mathcal{L}_x\colon \Omega^1(M) \to \Omega^1(M)$ is a derivation on T^*M. The linear vector field on T^*M is just the Hamiltonian vector field of the linear function $L_x \in C^\infty(T^*M)$ with respect to the canonical symplectic structure on T^*M.

2.3. **Covariant derivatives.** To finish this section, it is recalled the notion of covariant derivative (for more details, see [5]).

Definition 2.15. *A **covariant derivative** on a Lie algebroid* $(A \to M, \sharp, [\cdot,\cdot])$ *is a* $\mathbb{R}-$*bilinear map*

$$\begin{array}{rccc} \nabla: & \Gamma(A) \times \Gamma(A) & \to & \Gamma(A) \\ & (\alpha, \beta) & \mapsto & \nabla_\alpha \beta \end{array}$$

such that,

i) *It is* $\mathcal{C}^\infty(M)$*-linear in the first variable.*

ii)

$$\nabla_\alpha f\beta = f\nabla_\alpha\beta + \alpha^\sharp(f)\beta,\ f \in \mathcal{C}^\infty(M),\ \alpha, \beta \in \Gamma(A). \tag{2.13}$$

Remark 2.16. *Note that ii) in Definition 2.15 is equivalent to* $\nabla_\alpha \in Der(A)$*, for all* $\alpha \in \Gamma(A)$.

Using Eq. (2.11) for each $\alpha \in \Gamma(A)$, we can see $\nabla_\alpha \beta$ locally as follows,

$$(\nabla_\alpha \beta)_U = d\beta_{U|p}(\sharp_U(p, \alpha_U(p))) - D_U(p; \alpha_U(p), \beta_U(p)), \tag{2.14}$$

where $D_U : U \to L(F \times F, F)$ is a differentiable map.
Given a covariant derivative ∇ on A, the **torsion** T_∇ and the **curvature** R_∇ is the $(1,2)$-tensor field (respectively, $(1,3)$-tensor field).

$$\begin{aligned} T_\nabla(\alpha, \beta) &= \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta], \\ R_\nabla(\alpha, \beta), \gamma) &= \nabla_\alpha \nabla_\beta \gamma - \nabla_\beta \nabla_\alpha \gamma - \nabla_{[\alpha,\beta]} \gamma, \end{aligned}$$

for $\alpha, \beta, \gamma \in \Gamma(A)$. ∇ is said to be **torsionless** if $T_\nabla \equiv 0$ and **flat** if $R_\nabla = 0$.

Let $\gamma : I \to M$ be a $\mathcal{C}^\infty$ curve. By a **lift** a of γ to A we mean a $\mathcal{C}^\infty$ curve $a : I \to A$ such that

$$\pi \circ a = \gamma.$$

The space of these curves is denoted by $Lift(\gamma)$. Now, let $a : I \to A$ be an $A-$path and $\gamma = \pi \circ a$ the base path. Then, there is a **parallel transport**

$$\nabla_a : Lift(\gamma) \to Lift(\gamma),$$

such that, for each $\Lambda \in Lift(\gamma)$

$$\nabla_a \Lambda = \nabla_{\alpha_a} \alpha_\Lambda \circ \gamma,$$

where $\alpha_a, \alpha_\Lambda \in \Gamma(A)$ satisfy that

$$\alpha_a \circ \gamma = a \ ; \ \alpha_\Lambda \circ \gamma = \Lambda.$$

Writing $\nabla_{\alpha_a} \alpha_\Lambda$ locally we have:

$$(\nabla_a \Lambda)_U(t) = \frac{\partial}{\partial t} \Lambda_U - D_U(\gamma_U(t); a_U(t), \Lambda_U(t)). \tag{2.15}$$

Finally, an important concept associated with a covariant derivative is that of geodesic.

Definition 2.17. *Let $a : I \to A$ be an $A-$path. Then, a is a* ***geodesic*** *if*

$$\nabla_a a = 0.$$

Using Eq. (2.15), an $A-$path $a : I \to A$ is a geodesic if and only if locally

$$\frac{d}{dt} a_U = D_U(\gamma_U(t); a_U(t), a_U(t)). \tag{2.16}$$

3. Spray and covariant derivative

In this section, the notion of Lie algebroid spray will be introduced and its relation with covariant derivatives will be studied.

Definition 3.1. *$\mathcal{F} \in \mathfrak{X}(A)$ is said to be a **second-order differential equation (SODE)** on A if*

$$T\pi \circ \mathcal{F} = \sharp. \tag{3.17}$$

Note that, we can express $\mathcal{F}$ locally as the following map:

$$\begin{array}{rccl} \mathcal{F}_U: & U \times F & \to & \mathbb{R}^n \times F \\ & (p,v) & \mapsto & (g^1_U(p,v), f^1_U(p,v)) \end{array} \tag{3.18}$$

So, Eq. (3.17) is equivalent to $g^1_U(p,v) = \bar{\sharp}_U(p,v)$ for all $(p,v) \in U \times F$.

A SODE can be characterized in terms of its integral curves.

Proposition 3.2. *Let $\mathcal{F}$ be a vector field on A. $\mathcal{F}$ is a SODE on A if and only if every integral curve $a\colon I \to A$ of $\mathcal{F}$ is an $A-$path.*

Proof. If $a\colon I \to A$ is integral curve of $\mathcal{F}$, $\mathcal{F}(a(t)) = \frac{d}{dt}(a(t))$. Using Eq. (3.17),

$$\frac{d}{dt}(\pi \circ a) = T\pi\Big(\frac{d}{dt}(a(t))\Big) = (T\pi \circ \mathcal{F})(a(t)) = \sharp(a(t)).$$

The other implication can be proved similarly. □

We are interested in a special kind of second-order differential equations.

Definition 3.3. *Let $\mathcal{F} \in \mathfrak{X}(A)$ be a SODE on A. $\mathcal{F}$ is said to be a **spray** if for all $s \in \mathbb{R}$*

$$\mathcal{F}(s \cdot v) = Ts(s \cdot \mathcal{F}(v)), \ \forall v \in A. \tag{3.19}$$

Remark 3.4. *Let (M, Π) be a Poisson manifold and $T^*M \to M$ the associated Lie algebroid. Then, a spray of T^*M is just a **Poisson spray** [4]. In that paper, the authors give a simple proof of the existence of symplectic realizations using Poisson sprays. This result have been generalized in [1], where Lie algebroid sprays have been used to obtain an explicit and direct construction of a local Lie groupoid integrating a given Lie algebroid.*

Using the expression (3.18), $\mathcal{F}$ is a spray iff for all $s \in \mathbb{R}$,

$$f^1_U(p, s \cdot v) = s^2 \cdot f^1_U(p,v), \ \forall (p,v) \in U \times F.$$

Thus condition (3.19) (in addition to being a second-order vector field) simply means that f^1_U is homogeneous of degree 2 in the second variable. Hence, it follows that f^1_U is a quadratic map in its second variable, and specifically, this quadratic map is given by

$$f^1_U(p,v) = \frac{1}{2} d^2 f^1_{U,p|0}(v,v),$$

where, $f^1_{U,p}(v) = f^1_U(p,v)$, for any $(p,v) \in U \times F$.
Therefore, the spray is induced by a symmetric bilinear map given at each point p in a chart by

$$B_U(p) = \frac{1}{2} d^2 f^1_{U,p|0}.$$

Conversely, every differential map

$$B_U : U \to L_{SYM}(F \times F, F),$$

defines a spray over U. We call B_U **the symmetric bilinear map associated with the spray**. Now, motivated by Eq. (2.14), we introduce the following definition.

Definition 3.5. *Given a spray $\mathcal{F}$ and a covariant derivative ∇ over the same Lie algebroid A. We say that $\mathcal{F}$ is the spray associated with ∇ if ∇ can be written locally as follows:*

$$(\nabla_\gamma \beta)_U(p) = d\beta_{U|p}(\bar{\sharp}_U(p, \gamma_U(p))) - D_U(p; \gamma_U(p), \beta_U(p)),$$

where, for all $(p,v) \in U \times F$,

$$\mathcal{F}_U(p,v) = (\bar{\sharp}_U(p,v), D_U(p;v,v)).$$

Observe that, if $\mathcal{F}$ is associated with ∇, $\mathcal{F}$ is unique.
Let ∇ be a covariant derivative on A and $\gamma \in \Gamma(A)$. Then $\nabla_\gamma \in Der(A)$ and we can take the associated linear vector field on A^*, $(\xi^*_{\nabla_\gamma}, \gamma^\sharp)$, i.e.,

i) For each $\beta \in \Gamma(A)$,

$$\xi^*_{\nabla_\gamma}(L_\beta) = L_{\nabla_\gamma \beta}.$$

ii) For all $f \in \mathcal{C}^\infty(M)$,

$$\xi^*_{\nabla_\gamma}(f \circ \pi^*) = \gamma^\sharp(f) \circ \pi^*.$$

So, using Eq (2.11)

$$(\nabla_\gamma \beta)_U(p) = d(\beta_U)_{|p}(\gamma^\sharp_U(p)) - \xi_{\nabla_\gamma}{}^1_U(p, \beta_U(p)). \tag{3.20}$$

Hence, we can define $\mathcal{F} \in \mathfrak{X}(A)$ such that

$$\mathcal{F}(\gamma(p)) = \xi_{\nabla_\gamma}(\gamma(p)) \tag{3.21}$$

Where $(\xi_{\nabla_\gamma}, \gamma^\sharp)$ is the linear vector field on A associated with $(\xi^*_{\nabla_\gamma}, \gamma^\sharp)$.
Note that $\nabla_\gamma(p)$ depends only on the value of $\gamma(p)$ so we can define the differentiable map $B_U : U \to L_{SYM}(F \times F, F)$ given by

$$B_U(p; v, w) = \frac{1}{2}(\xi_{\nabla_{\gamma_v}}(\gamma_w(p)) + \xi_{\nabla_{\gamma_w}}(\gamma_v(p))),$$

where $\gamma_v(p) = v$ and $\gamma_w(p) = w$. Let us see it is well-defined, i.e., for all $v, w \in F$,

$$B_U(p; v, w) \in L_{SYM}(F \times F, F).$$

Indeed, taking $\gamma_v, \gamma_w \in \Gamma(A)$ as above we only have to use that ∇ is $\mathbb{R}-$linear in the first variable, that ξ_{∇_γ} is a linear vector field and the following equality:

$$\xi_{\lambda\nabla_\gamma + \nu\nabla_{\gamma'}} = \lambda\xi_{\nabla_\gamma} + \nu\xi_{\nabla_{\gamma'}}.$$

Finally, it is obvious that $B_U(p)$ is a symmetric map and $f^1_U(p,v) = B_U(p;v,v)$. Now, using Eq (3.21),

$$T_{\gamma(p)}\pi(\mathcal{F}(\gamma(p))) = T_{\gamma(p)}\pi(\xi_{\nabla_\gamma}(\gamma(p))) = \gamma^\sharp(p).$$

Therefore, $\mathcal{F}$ is a spray.
Now, let $\mathcal{F}$ be a spray on A and $T : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ a $(1,2)-$tensor field on A. Construct a covariant derivative ∇ such that

i) For each $\alpha \in \Gamma(A)$

$$\xi_{\nabla_\alpha} \circ \alpha = \mathcal{F} \circ \alpha,$$

where ξ_{∇_α} is the linear vector field on A associated with ∇_α.

ii) For all $\alpha, \beta \in \Gamma(A)$,

$$T(\alpha,\beta) = T_\nabla(\alpha,\beta) = \nabla_\alpha\beta - \nabla_\beta\alpha - [\alpha,\beta].$$

Hence

$$\nabla_\alpha\beta = \frac{1}{2}\{\nabla_{\alpha+\beta}(\alpha+\beta) - \nabla_\alpha\alpha - \nabla_\beta\beta + T(\alpha,\beta) + [\alpha,\beta]\}. \tag{3.22}$$

So, using Eq (3.20), locally

$$\begin{aligned}(\nabla_\alpha\beta)_U(p) \;=\; & \frac{1}{2}\Big\{ d\beta_{U|p}(\alpha^\sharp_U(p)) + d\alpha_{U|p}(\beta^\sharp_U(p)) + f^1_U(p, \beta_U(p) + \alpha_U(p)) \\ & -f^1_U(p,\beta_U(p)) - f^1_U(p,\alpha_U(p)) + T(\alpha,\beta)_U(p) + [\alpha,\beta]_U(p)\Big\}.\end{aligned}$$

Thus, it has been proved the uniqueness and the smoothness of $\nabla_\alpha\beta$, for all $\alpha, \beta \in \Gamma(A)$. Finally, using this expression, we can prove that ∇ is, indeed, a covariant derivative on A.
The previous discussion can be summarized in the following theorem.

Theorem 3.6. *Every covariant derivative on A has an unique associated spray on A. Furthermore, given a spray on A, $\mathcal{F}$, and a $(1,2)-$tensor field on A, T, there exists a unique covariant derivative ∇ such that $\mathcal{F}$ is the associated spray with ∇ and*

$$T_\nabla = T.$$

Taking $\mathcal{F}$ the spray associated with ∇, a is a geodesic iff

$$\frac{d}{dt}a = \mathcal{F}(a(t)),$$

i.e., a is an integral curve of $\mathcal{F}$. Since every integral curve of $\mathcal{F}$ is an $A-$path (Proposition 3.2),

Proposition 3.7. *Let $a : I \to A$ be a $\mathcal{C}^\infty$ curve. a is geodesic iff is integral curve of $\mathcal{F}$.*

Example 3.8. Let $A \to M$ be a Lie algebroid and assume that g is a fibered metric on A, that is, for each $p \in M$ $g(p)$ is a nondegenerate symmetric bilinear form on A_p. It is clear that g can be seen as a $\mathcal{C}^\infty(M)$-linear map

$$g : \Gamma(A) \times \Gamma(A) \to \mathcal{C}^\infty(M).$$

This map allows to define a $\mathcal{C}^\infty(M)-$isomorphism $b_g : \Gamma(A) \to \Gamma(A^*)$, such that, for all $\alpha, \beta \in \Gamma(A)$

$$\{b_g(\alpha)\}(\beta) = g(\alpha, \beta).$$

There exists a unique torsionless covariant derivative ∇ on A such that for all sections α, β, γ we have

$$\alpha^\sharp(g(\beta,\gamma)) = g(\nabla_\alpha \beta, \gamma) + g(\beta, \nabla_\alpha \gamma).$$

This covariant derivative is called the **Levi-Civita derivative of g** and is denoted by ∇^g. The Levi-Civita derivative is characterized by the **Koszul formula**

$$\begin{aligned} g(\nabla^g_\beta \alpha, \gamma) &= \frac{1}{2}\{\alpha^\sharp(g(\beta,\gamma)) + \beta^\sharp(g(\gamma,\alpha)) - \gamma^\sharp(g(\alpha,\beta)) - \\ &\quad - g([\alpha,\beta],\gamma) - g([\alpha,\gamma],\beta) - g([\beta,\gamma],\alpha)\}, \end{aligned} \tag{3.23}$$

The spray associated with ∇^g will be called **metric spray**.

4. ∇ INFINITESIMAL AUTOMORPHISMS

Throughout this section, we fix a covariant derivative ∇ on a Lie algebroid $(A \to M, \sharp, [\cdot,\cdot])$, with $\mathcal{F} \in \mathfrak{X}(A)$ as the spray associated with ∇.

Definition 4.1. *Let $\pi : A \to M$ be a Lie algebroid with anchor $\sharp : A \to M$. Given a covariant derivative ∇ on A, a vector bundle isomorphism (Φ, ϕ) on A is a $\nabla-$**automorphism** if*

$$\Phi_*(\nabla_\alpha \beta) = \nabla_{\Phi_*\alpha} \Phi_* \beta, \qquad \alpha, \beta \in \Gamma(A). \tag{4.24}$$

Let (Φ, ϕ) be a ∇-automorphism. Then, as a first consequence, we have that

$$\sharp \circ \Phi = T\phi \circ \sharp. \tag{4.25}$$

Indeed, if $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$

$$\Phi_*\Big(\nabla_\alpha(f\beta)\Big) = \nabla_{\Phi_*\alpha}\Phi_*(f\beta).$$

Using Eq. (2.13), it is deduced that $\gamma^\sharp(f) \circ \phi = (\Phi_*\gamma)^\sharp(f \circ \phi)$ or, equivalently, (4.25) holds.

Proposition 4.2. *Let (Φ, ϕ) be a vector bundle automorphism. Then, (Φ, ϕ) is a $\nabla-$automorphism if and only if for each $\alpha \in \Gamma(A)$,*

$$\xi_{\nabla_{\Phi_*\alpha}} = (T\Phi)_*(\xi_{\nabla_\alpha}), \tag{4.26}$$

where ξ_{∇_α} is the linear vector field on A associated with ∇_α.

Proof. First, we have that

$$\xi^*_{\nabla_{\Phi_*\alpha}} = ((T\Phi^T)^{-1})_*(\xi^*_{\nabla_\alpha}). \tag{4.27}$$

Indeed, given $\beta \in \Gamma(A)$, from the fact that $L_\beta \circ \Phi^T = L_{\Phi_*^{-1}\beta}$,

$$\begin{aligned} ((T\Phi^T)^{-1})_*(\xi^*_{\nabla_\alpha})(L_\beta) &= (\xi^*_{\nabla_\alpha})(L_\beta \circ (\Phi^T)) \circ (\Phi^T)^{-1} = (\xi^*_{\nabla_\alpha})(L_{\Phi_*^{-1}\beta}) \circ (\Phi^T)^{-1} \\ &= (L_{\nabla_\alpha \Phi_*^{-1}\beta}) \circ (\Phi^T)^{-1} = L_{\Phi_*(\nabla_\alpha \Phi_*^{-1}\beta)} = L_{\nabla_{\Phi_*\alpha}\beta} \\ &= \xi^*_{\nabla_{\Phi_*\alpha}}(L_\beta). \end{aligned}$$

Using that if ξ is linear vector field with flow $\{(\Phi_t, \phi_t)\}$ then the flow of ξ^* is $\{\Phi^T_{-t}\}$ (see Remark 2.9). Thus, for any (F, f) vector bundle isomorphism, the relation $((TF)_*\xi)^* = ((TF^{-1})^T)_*\xi^*$ holds. Therefore, Eq. (4.27) is equivalent to

$$\xi^*_{\nabla_{\Phi_*\alpha}} = ((T\Phi)_*(\xi_{\nabla_\alpha}))^*.$$

But this relation is equivalent to Eq. (4.26), since two linear vector fields ξ_1 and ξ_2 are equal if and only if ξ_1^* and ξ_2^* coincide.
Conversely, suppose that Eq. (4.26) is satisfied. Then, for all $\beta \in \Gamma(A)$,

$$((T\Phi^T)^{-1})_*(\xi^*_{\nabla_\alpha})(L_\beta) = \xi^*_{\nabla_{\Phi_*\alpha}}(L_\beta).$$

Taking into account that

$$((T\Phi^T)^{-1})_*(\xi^*_{\nabla_\alpha})(L_\beta) = L_{\Phi_*(\nabla_\alpha \Phi_*^{-1}\beta)},$$

we have

$$\Phi_*(\nabla_\alpha \Phi_*^{-1}\beta) = \nabla_{\Phi_*\alpha}\beta.$$

□

Corollary 4.3. *Let (Φ, ϕ) be an $\nabla-$automorphism. Then $\mathcal{F}$ is $(\Phi, \phi)-$invariant, i.e.,*

$$(T\Phi)_*\mathcal{F} = \mathcal{F}.$$

Proof. Using the previous proposition,

$$(T\Phi)_*(\xi_{\nabla_{\Phi_*^{-1}\alpha}}) = \xi_{\nabla_\alpha}, \qquad \alpha \in \Gamma(A). \tag{4.28}$$

Then, given $p \in M$,

$$\begin{aligned} \mathcal{F}(\alpha(p)) &= \xi_{\nabla_\alpha}(\alpha(p)) = T\Phi^{-1}\{\xi_{\nabla_{\Phi_*^{-1}\alpha}}(\Phi(\alpha(p)))\} \\ &= T\Phi^{-1}\{\xi_{\nabla_{\Phi_*^{-1}\alpha}}(\Phi_*^{-1}\alpha(\phi(p)))\} = T\Phi^{-1}\{\mathcal{F}(\Phi_*^{-1}\alpha(\phi(p)))\} \\ &= (T\Phi)_*\mathcal{F}(\alpha(p)). \end{aligned}$$

□

Definition 4.4. *Let (ξ, x) be a linear vector field on A with (local) flow $\{(\Phi_t, \phi_t)\}$. Then, (ξ, x) is said to be a ∇* ***infinitesimal automorphism*** *(∇ i.a.) if (Φ_t, ϕ_t) is a (local) $\nabla-$automorphism for all t, i.e.,*

$$\Phi_{t*}(\nabla_\alpha \beta) = \nabla_{\Phi_{t*}\alpha}\Phi_{t*}\beta, \tag{4.29}$$

for all $\alpha, \beta \in \Gamma(A)$.

Now, we shall give an equivalent condition defining a ∇ infinitesimal automorphism in terms of the corresponding derivation.

Proposition 4.5. *Let (ξ, x) be a linear vector field on A^*. The following properties are equivalent.*

i) *(ξ, x) is a ∇ infinitesimal automorphism.*
ii) *For all $\alpha, \beta \in \Gamma(A)$,*

$$D_{\xi^*}\nabla_\alpha\beta = \nabla_{D_{\xi^*}\alpha}\beta + \nabla_\alpha D_{\xi^*}\beta. \tag{4.30}$$

Proof. Taking derivatives in Eq. (4.29) and using Proposition 2.10, it is clear that (4.30) is equivalent to

$$\frac{d}{dt}_{|t=0}(\Phi_{t*}(\nabla_\alpha\beta)) = \frac{d}{dt}_{|t=0}(\nabla_{\Phi_{t*}\alpha}\Phi_{t*}\beta).$$

From this relation, $i)$ implies $ii)$.
Conversely, noting that (4.29) is equivalent to

$$\nabla_\alpha\beta = \Phi_{-t*}\Big(\nabla_{\Phi_{t*}\alpha}\Phi_{t*}\beta\Big),$$

we only have to prove that

$$\frac{d}{dt}_{|t=s}(\Phi_{-t*}(\nabla_{\Phi_{t*}\alpha}\Phi_{t*}\beta)) = 0.$$

For $s = 0$ it is true. Now, for a general s,

$$\begin{aligned}\frac{d}{dt}_{|t=s}\Big(\Phi_{-t*}(\nabla_{\Phi_{t*}\alpha}\Phi_{t*}\beta)\Big) &= \frac{d}{dt}_{|t=0}\Big(\Phi_{-t-s*}(\nabla_{\Phi_{t+s*}\alpha}\Phi_{t+s*}\beta)\Big)\\ &= \Phi_{-s*}\Big(\frac{d}{dt}_{|t=0}(\Phi_{-t*}(\nabla_{\Phi_{t*}(\Phi_{s*}\alpha)}\Phi_{t*}(\Phi_{s*}\beta)))\Big)\\ &= 0.\end{aligned}$$

□

Corollary 4.6. *Let (ξ, x) be a linear vector field on A. (ξ, x) is a ∇ infinitesimal automorphism if and only if*

$$[\xi, \xi_{\nabla_\alpha}] = \xi_{\nabla_{D_{\xi^*}\alpha}}, \ \forall \alpha \in \Gamma(A). \tag{4.31}$$

Proof. Using Proposition 4.2 for the (local) flow of (ξ, x), we only have to prove that Eq. (4.26) is equivalent to Eq. (4.31).
First, using Proposition 2.10,

$$[\xi, \xi_{\nabla_\alpha}] = \frac{d}{dt}_{|t=0}((T\Phi_t)_* \xi_{\nabla_\alpha}) = \xi_{\nabla_{\frac{d}{dt}_{|t=0}(\Phi_{t*}\alpha)}} = \xi_{\nabla_{D_{\xi^*}\alpha}}.$$

Conversely, Eq. (4.31) implies that

$$\frac{d}{dt}_{|t=0}((T\Phi_t)_* \xi_{\nabla_\alpha}) = \xi_{\nabla_{\frac{d}{dt}_{|t=0}(\Phi_{t*}\alpha)}}.$$

Then,

$$\frac{d}{dt}_{|t=0}((T\Phi_{-t})_* \xi_{\Phi_{t*}\nabla_\alpha}) = 0.$$

So, for each s

$$\frac{d}{dt}_{|t=s}((T\Phi_{-t})_* \xi_{\nabla_{\Phi_{t*}\alpha}}) = (T\Phi_{-s})_* \frac{d}{dt}_{|t=0}((T\Phi_{-t})_* \xi_{\nabla_{\Phi_{t*}\Phi_{s*}\alpha}}) = 0.$$

Hence, for all t

$$(T\Phi_{-t})_* \xi_{\nabla_{\Phi_{t*}\alpha}} = \xi_{\nabla_\alpha},$$

i.e.,

$$\xi_{\nabla_{\Phi_{t*}\alpha}} = (T\Phi_t)_* \xi_{\nabla_\alpha}.$$

□

From Corollary 4.3, we deduce the following result.

Theorem 4.7. *If (ξ, x) is a ∇ infinitesimal automorphism then*

$$[\xi, \mathcal{F}] = 0.$$

Remark 4.8. Note that the relation

$$[\xi, \mathcal{F}] = 0,$$

implies that

i) The relation $\sharp \circ \Phi = T\phi \circ \sharp$ holds.
ii) For all $\alpha \in \Gamma(A)$, $\nabla_{\Phi_*\alpha} \Phi_*\alpha = \Phi_* \nabla_\alpha \alpha$.

Thus, using Eq (3.22), we can prove the reciprocal only for special cases such as ∇ symmetric or (Φ, ϕ) Lie algebroid automorphism with $T_\nabla \equiv 0$.

An important particular case of ∇ infinitesimal automorphism is the following one.

Definition 4.9. *Let α be a section of $(A \to M, \sharp, [\cdot, \cdot])$ with flow $\{(\Phi_t, \phi_t)\}$. Then, α is said to be a ∇* ***infinitesimal inner automorphism*** *if the flow (Φ_t, ϕ_t) is a $\nabla-$isomorphism for all t.*

Observe that, taking into account that

$$[\mathcal{H}_{L_\alpha}, \mathcal{H}_{L_\beta}] = \mathcal{H}_{L_{[\alpha,\beta]}},$$

the space of ∇ infinitesimal inner automorphisms is a Lie subalgebra of $(\Gamma(A), [\cdot,\cdot])$. From Example 2.12, Proposition 4.5 can be written for this particular case as follows.

Proposition 4.10. *Let $(A \to M, \sharp, [\cdot,\cdot])$ be a Lie algebroid and $\alpha \in \Gamma(A)$. The following properties are equivalent.*

i) *α is a ∇ infinitesimal inner automorphism.*
ii) *For all $\gamma, \beta \in \Gamma(A)$,*

$$[\alpha, \nabla_\gamma \beta] = \nabla_{[\alpha,\gamma]}\beta + \nabla_\gamma[\alpha,\beta]. \tag{4.32}$$

Recall that (see Example 2.12) the flow of α is just the flow of the Hamiltonian $(\mathcal{H}_{L_\alpha}, \alpha^\sharp)$ where

$$\mathcal{H}_{L_\alpha}(L_\beta) = L_{[\alpha,\beta]},\ \forall \beta \in \Gamma(A),$$

i.e.

$$D_{\mathcal{H}_{L_\alpha}}\beta = [\alpha,\beta]. \tag{4.33}$$

From this relation and taking into account that the flow of a section is a (local) Lie algebroid isomorphism, the following theorem is a particular case of Corollary 4.6.

Proposition 4.11. *Let α be a section of $(A \to M, \sharp, [\cdot,\cdot])$. If α is ∇ infinitesimal inner automorphism if and only if for all $\beta \in \Gamma(A)$,*

$$[\alpha^c, \xi_{\nabla_\beta}] = \xi_{\nabla_{[\alpha,\beta]}}.$$

Thus, from Theorem 4.7, Remark 4.8 and the fact that the flow of a section is (local) Lie algebroid automorphism, we have.

Theorem 4.12. *Let $(A \to M, \sharp, [\cdot,\cdot])$ be a Lie algebroid, ∇ a covariant derivative and $\alpha \in \Gamma(A)$. If α is a ∇ infinitesimal inner automorphism then*

$$[\alpha^c, \mathcal{F}] = 0.$$

Conversely, if ∇ is torsionless or symmetric $[\alpha^c, \mathcal{F}] = 0$ implies that α is a ∇ infinitesimal inner automorphism.

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