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THE

PHILOSOPHICAL TRANSACTIONS AND COLLECTIONS,

To the End of the Year MDCC,

ABRIDGED,

AND

Dispofed under GENERAL HEADS.

VOLUME I.

Containing all the

MATHEMATICAL PAPERS.

By *JOHN LOWTHORP*, M.A. and F.R.S.

The FIFTH EDITION, Corrected,
In which the LATIN Papers are now first tranflated into ENGLISH.

LONDON,
Printed for W. INNYS, R. WARE, J. and P. KNAPTON, S. BIRT, E. COMYNS, D. BROWNE, T. LONGMAN, H. WHITRIDGE, C. HITCH, J. HODGES, S. AUSTEN, R. MANBY and H. S. COX, A. MILLAR, J. and J. RIVINGTON, and J. WARD.

MDCCXLIX.

To the HONOURABLE

Sir *ISAAC NEWTON*, Kt:

PRESIDENT;

And to the

COUNCIL and FELLOWS

OF THE

Royal Society of London

For the Advancement of

NATURAL KNOWLEDGE,

THESE

MATHEMATICAL PAPERS,

Abridged and Diſpoſed under GENERAL HEADS,

Are moſt humbly Dedicated, by

John Lowthorp.

THE

PREFACE.

THE Philoſophical Tranſactions *having met with* general Applauſe *and* Encouragement *for many Years, it would be a* needleſs Trouble *to give any* Hiſtory *of them: 'Tis enough to ſay, that many of the* Diſcourſes *were* compoſed, *and all of them* collected *and* publiſhed *by* particular Members *of the* Royal Society. *I ſhall therefore employ theſe very few Pages, only to acquaint the* Reader *with my own* Conduct *in this* Abridgment *of them.*

When I firſt reſolved *upon this* Undertaking, *I had two Sorts of* Readers *in view, whom I was* deſirous *to ſerve*; thoſe who make uſe of Books for their private Inſtruction or Entertainment, and thoſe who conſult them in order to publiſh ſomething of their own. *To a* Reader *of the former Claſs, I thought it ſufficient to give him the Subſtance of ſo many* curious Papers, *in ſuch* Order *as would beſt ſuit with the* Courſe *of thoſe* Studies *that might denominate him a* general Scholar. *But, for the Sake of the latter, I have, in the Margin, given the* Title *and* Author *of each Paper, and directed to the* Number *and* Page *of the* Tranſactions *or* Collections, *where he may meet with the* Original *itſelf. To the former, I* deſigned *this* Abridgment *to be as uſeful as the* Volumes *at large; and to ſerve the latter, inſtead of a not* inconvenient Repertorium.: *And in the Proſecution of this Deſign, I have generally* confined *myſelf to theſe Rules.*

I. *I have not only* retained *the Eſſential Parts of the* Diſcourſes, *but I have kept in many Places to the very* Words *of their own* Authors, *(except where I was* forced *to vary them a little, to preſerve the* Connection:) *For, I thought it very unwarrantable to* obtrude *any thing of mine under the Name of another Perſon.*

II. *But to* ſhorten *the* whole Work, *wherever I found any* Perſonal Addreſſes, *long and unneceſſary* Excurſions, *or* pompous Citations *of* Books, *I have taken the Liberty to* ſuppreſs *them*; *yet, I hope, without injuring the* Force *of the* Author's Reaſoning.

III. *I have* omitted *all* Accounts *and* Extracts *of* Books, *which now, after ſo many Years* Publication, *ſeem almoſt uſeleſs: Yet, to put the* Readers in mind *of them, eſpecially ſuch as are about to furniſh or enlarge their* Libraries, *I have* added *a* Catalogue *at the* End *of each* Chapter *to which they chiefly belong*; *and I have alſo* directed *them to ſuch* Additions, Emendations, *or* Refutations, *as ought to be* conſulted, *when thoſe Books fall under their* Examination.

IV. *I have alſo* omitted *all* Heads *of* Inquiries *and* Experiments *ſimply propoſed, without farther* Proſecution; *believing that the* Anſwers *already given to many of them, and other* Diſcourſes *upon the ſame, or the like* Subjects, *will ſufficiently direct the* Notice *of an* Inquiſitive Reader.

V. *The previous* Calculations *of* Eclipſes, Lunar Appulſes, *and* Satellite Eclipſes *and* Occultations; *alſo* Tide-Tables, *and many other curious* Papers *of that kind, have long ago outlived the Reaſon of their* Publication.

VI. *All Simple* Catalogues *of* Natural Curioſities *(as of* Shells, Minerals, Plants, Animals, *&c.) without particular* Deſcriptions *of them, are little* inſtructive: *and chiefly*

ſerve

ſerve to enlarge *the* Hiſtory *of the* Muſeum, *where they are* depoſited: *Which is no Part of the* Deſign *of theſe* Volumes.

VII. *I have commonly* omitted *ſuch Papers as have been* collected *into juſt* Volumes *by their own* Authors. *For this Reaſon I have* omitted *ſome of thoſe ſurprizing* Microſcopial Diſcoveries *by the* famous *M.* Leeuwenhoeck: *But I farther confeſs, I was alſo leſs inclined to inſert them here, becauſe moſt of them treat of* Subjects *not at all convenient* (in my Opinion) *for* common Readers.

VIII. *But, to do all the* Right *I could to the* ingenious Authors *of thoſe* Papers, *which the* Limits *of this* Abridgment *obliged me to* omit, *I have, at the* End *of each* Chapter, *annexed their* Titles, *and ſometimes a ſhort* Account *of them.*

Theſe are the Rules *I have carefully obſerved thro' the whole Conduct of this* tedious Work: *Wherein I have faithfully aimed at the* General Good *of all ſorts of* Readers; *if I have failed in the* Performance, *'tis for want of* Judgment *to do it* better: *But I am bold to ſay, That if a kind* Reception *of this ſhall encourage a like* Abridgment *of the* Foreign Philoſophical Journals, *in the* ſame *or a* better Order, *it will much facilitate the many* Diſcoveries *ſtill ready to* reward *the* Labours *and* Expences *of all* induſtrious Promoters *of* Natural Knowledge.

Adver-

Advertiſement of the Bookſellers

TO THE

FIFTH EDITION.

THE *Philoſophical Tranſactions*, ſince the Year 1700, having been abridged, and diſpoſed under General Heads, (after the ſame Method with this Work) by Mr. *JONES*, and by him publiſhed in Two Volumes: We thought it would tend to make the whole complete, if the ſeveral *Indices* of the Five Volumes were made into One; by which means the Reader will have but one Trouble in ſeeking any Particular he has occaſion for. To the ſame Purpoſe we have thrown the *Contents* of the ſeveral Volumes together, and prefixed them to the Firſt.

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2. *The*

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2.—*on*

2.—*the*

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OF THE
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WHEREIN

The *Numeral Letters* diſtinguiſh the *Volumes*, ii. denotes the ſecond Part, (in the IVth and Vth Vols.) and the *Figures* ſignify the *Pages*: The Numbers following immediately after the *Names*, or without any *Marks*, direct to the *Papers Abridg'd* and *Inſerted*; thoſe after *, to the *Titles* of *Books*; and thoſe after **, to the *Titles* of the *Papers Omitte d*

Patrick

C.

D.

E.

F.

Mr. John

Mr. John

J.

K.

L.

M.

S.

M. Du.

M. Pitton

V.

W.

THE

Mathematical Papers,

PUBLISH'D and DISPERS'D

IN THE

Philosophical Transactions

AND

COLLECTIONS;

ABRIDG'D,

AND

Dispos'd under GENERAL HEADS.

CHAP. I.

Geometry, Algebra, Arithmetick, Logarithmotechny.

I. 1. AS to what I formerly consider'd, about the Improvement of the Mathematical Sciences, the Result was chiefly this: While Men are destitute of Inclination, Genius, Assistances, and Leisure necessary for these Studies, no wonder if they make no greater Progress in them. Therefore it seems probable to me, that by the Help of the following Means, a tolerable good Remedy may be found for this Evil. That is, if, *Dr. J. Pell's Idea of the Mathematicks. An. 1638. Phil. Col. N. 5. p. 127.*

§. 1. A Mathematical Monitor (as we may call it) be compos'd, which may give proper Answers to these three Questions. 1. What Advantages and of

 what

what Kind, may be expected from the Study of Mathematicks? 2. What Helps are now in being for attaining ſo advantageous a Knowledge? 3. What Order is to be obſerved in making uſe of thoſe Aſſiſtances? Therefore this Monitor ſhould contain,

1. An eaſy and perſpicuous Diſcourſe upon the Limits or Extent of the Mathematical Arts, and of the conſiderable Advantages that will accrue, not only to the Perſons themſelves that ſtudy them, but likewiſe to a Nation that abounds in ſkilful Mathematicians.

2. A Catalogue of Mathematicians, and of Works publiſh'd by them; which is to exhibit, 1. A Synopſis of all kinds of Mathematical Books, whether ſuch as are already publiſh'd, or ſuch as are yet unpubliſh'd, and, being in Manuſcript, lie concealed in publick Libraries; proper Numbers or References being affix'd to every Kind. 2. A Chronological Catalogue of all the celebrated Mathematicians, diſpoſed according to the Ages in which they flouriſhed; always ſubjoining the Year of our Lord in which their Works were firſt printed. 3. A Catalogue of the ſame Works, according to the Series of Years, in which they were printed in any Language. In digeſting of which I would proceed in ſuch Manner, that marking the Year of our Lord, I would add (as in common Catalogues) the Names of all the Mathematical Books that were publiſh'd that Year, in any Country or any Language; 1. Shewing in each how much the Volume contain'd, by marking not only whether it was in Folio, Quarto, *&c.* but the whole Number of Pages, ſo that the Bulk of the Work might eaſily be known. 2. Before the Title mentioning the Year, to which any one might turn back, who ſhould deſire to know when the Book was wrote, and when it was laſt publiſh'd in any Language. 3. Marking in the Margin after the Title, 1. The Year in which any Work was laſt printed. 2. The Number referring the Reader to the Synopſis, which was given in the firſt Page of the Catalogue. Now by the Help of theſe Numbers any one might eaſily and readily run through all the Mathematical Books belonging to one Subject.

3. An Admonition to the Studious, which are the beſt Books in every Kind, in what Order and Method they are to be read, what is to be choſen and what omitted in reading ſome of the minor Mathematicians, how we are to proceed ſo as to retain every Thing in Memory.

4. An Exhortation and Encouragement to all thoſe, who are ſufficiently provided with Wealth, Opportunity, and Ingenuity for the Purſuit of theſe Studies; that, 1. Having Regard to the great Advantages that redound from hence not only to themſelves but to all Mankind in general. 2. As likewiſe to that pure and ſincere Pleaſure which ariſes from the Search of hidden Truths, and from ſtriving with difficult Problems and the Conqueſt of them; that they may ſeriouſly apply themſelves to the Advancement of Science, and ſo much the rather, as, 3. More expeditious Methods are now found out than were known to our Anceſtors, which ſave us much Labour, Time, and Expence. Then an Exhortation to all ſuch as are eminent for ſetting a right Value on theſe Studies, and are likewiſe diſtinguiſhed for Power and Wealth (which ſurely may be made inſtrumental to perpetual Fame, if prudently diſpenſed) that they

may become Patrons to ingenious Men of this Kind, by propofing handfome Rewards to the moft deferving of them, to encourage them to complete fuch Difcoveries as their own Genius's may prompt them to. Laftly, To all Princes and Commonwealths, who cannot eafily procure a greater Ornament to their Dominions, than by making it their Endeavour, 1. That they may abound with Perfons fkilled in thefe Arts. 2. That the Way leading to them may be made as little laborious and expenfive as poffible. 3. That Mathematical Genius's may be more publickly known, and meet with fuch Encouragement as they fhall deferve.

For this end it will be very neceffary that,

§. 2. A Publick Library may be founded, which may be furnifh'd with all the Books abovementioned, and with one Inftrument of every Sort that has been yet invented; and befides may have an Endowment fufficient, 1. To purchafe Copies of all the Mathematical Books that fhall be yearly publifh'd any where abroad. 2. To maintain a Library-keeper, whofe Bufinefs it fhould be,

1. To read over all the Books of this Kind, which are publifh'd in his own Country. 1. Suppreffing thofe which are not wrote according to the Rules of Art, that their Miftakes may not lead their Readers into Error. 2. To admonifh Authors, left they fhould only republifh Things already known, and treated of by others.

2. On Peril of their Reputation that they fhould approve of notable Inventions, and heartily recommend the Inventors to proper Patrons.

3. To receive, to enter into their Catalogue, and difpofe in their proper Repofitories, one Copy of the Books fo read over; when prefented to the Library well bound up, at the Charge of the Author or Bookfeller.

4. To give a civil and ready Anfwer to any ftudious Perfon, who fhall confult him about any Problem, whether it is already folved or no; left he fhould attempt any Thing that is well done already, or on the contrary fupprefs his Difcoveries, out of Fear they may be already known, and perhaps difcuffed in fome of the Books of the Library.

5. To receive, &c. all Manufcripts that may be prefented to the Library, or bequeathed to it by Legacy.

6. To keep a conftant Literary Correfpondence with all Perfons of this Kind, that refide in foreign Countries; left he fhould be ignorant of what Books are publifh'd there.

7. To take Notice among his Countrymen, who are fitteft and moft expert in inftructing others in thefe Arts.

8. To have an Acquaintance with all kinds of Artificers, who excel in the conftructing of Mathematical Inftruments and Contrivances, whether they work in Wood, Loadftones, Metal, Glafs, &c.

9. After a fair Trial to give their Teftimony, both of fpeculative Knowledge and practical Dexterity, to practical Men of all Kinds, whether Mafters of Ships, Surveyors, Accomptants, &c. that fuch as have Occafion for this Kind of Men may not be impofed on by ignorant Pretenders, to their great Lofs.

The Catalogue will eafily inform, which in fuch a Multitude of Books that almoft overwhelm the World, belong only to this Kind of Study. The Library will exhibit a Copy of every fuch Book, and inform where more Copies may be bought. It will alfo be a Kind of Storehoufe both to Natives and Foreigners, whence they may eafily learn, what Affiftances that Country can fupply to thefe Studies.

And this, in my Opinion, is the readieft Way of making ufe of the Helps we are already in Poffeffion of. If more are wanting, it will be neceffary, that by the Affiftance of fkilful Artifts,

§. 3. The three following new Treatifes may be compofed and publifhed.

1. Mathematical Pandects, containing, as perfpicuoufly, methodically, compendioufly, and ingenioufly as can be done, whatever may be collected, or deduced by Way of Corollary, from the Mathematical Books or Difcoveries made before our Time; quoting the moft ancient Authors in which they are found, at the End of every Period or Propofition; and fo marking in all the following Authors where they have been caught in a Theft, or where they have borrow'd without making any Acknowledgment, or (what is worft of all) have boldly claimed to themfelves the Inventions of others. By this means that large Library would be contracted into a much narrower Compafs, to a great faving of Labour, Time, and Expence for thofe that come after; and this much more than any one would imagine at prefent. But now fince this Work would hardly make a portable Volume, there fhould be prepared alfo,

2. A Mathematical Companion, containing in a Manual (and therefore as concifely as may be) all the moft ufeful Tables, with Precepts to fhew their Application to folving of Problems, whether of pure Mathematicks, or applied to other Subjects.

Finally, that we may not always be confined to Books in this Kind of Learning, there fhould be contrived,

3. The Self-fufficient Mathematician, or an Inftruction to fhew how any Mathematician, who is no Enemy to Labour, may acquire fo much Skill, that without the Affiftance of Books or Inftruments he may attain the Solution of any Mathematical Problem, and that as eafily as another would folve it by turning over Books.

And this is that Idea of Mathematicks which, in my Manner, I have long ago figured to myfelf; being always firmly perfuaded, that then only we can hope for Affiftance in great Undertakings, when we have conceived an exact Idea of them in our Minds, and of the moft appofite Means of putting them in Execution. And if we cannot exprefs this Idea in fact, yet it is fomething to come as near it as may be. I imagine this is fo far from being above human Power, that I think the Induftry of one Man alone to be equal to it, who is not hindered by his own domeftick Affairs, or immerfed in a Multitude of bufy Cares. For it is evident that the Library and Catalogue may eafily be provided, if Money is not wanting. And as to the Pandects above defcribed, if the Tafk of compofing them were committed to me, I fhould impofe upon myfelf much feverer Conditions than I have mention'd there. For firft I would

delineate

delineate the infallible Proceſs of human Reaſon in the Inveſtigation of whatever it propoſes to itſelf, by ſhewing how it proceeds from the firſt Principles or Rudiments, by an uninterrupted Chain, to the moſt ſublime as well as the loweſt Application of them. Which Art perhaps Men would not be long without, if hereafter they ſhould carefully examine, by what Means ſuch Thoughts have aroſe in the Minds of certain Men whom they admire, how ſuch apt Means have been found out to attain ſuch an End. How theſe Pandects may be abridg'd into a Manual, ſuch as may be fit for common uſe, may not be difficult to underſtand. But ſo to fix them in their Minds, that they ſhall have no farther Need of Books (which is what is aimed at by our Self-ſufficient Mathematician) will be thought by moſt to exceed the Power of the human Mind; ſince no one that I know of has yet ventured to conceive ſuch a Thing in his Mind. Yet I believe that Men will diſmiſs ſomething of their Incredulity, when they conſider ſeriouſly with themſelves what Arts have been found out for ſtrengthening the Imagination, for aſſiſting the Memory, and for directing the Reaſoning Faculty, and what wonderful Effects may be produced by their Conjunction and conſtant Exerciſe.

2. It ſeems to me much more adviſeable, if inſtead of all that Apparatus, which the Author of the Idea propoſes, about collecting the various Writings of the Mathematicians, only the beſt and moſt deſerving were ſelected. And firſt thoſe ancient Authors ſhould be choſen, whoſe Works are ſtill extant; as *Euclid*, *Apollonius*, *Archimedes*, *Theodoſius*, *Pappus*, *Ptolemy*, with their other Fragments and Manuſcripts which have not yet ſeen the Light, ſome of which are in *Golius's* Cuſtody at *Leyden*, and ſome are preſerved at *Rome*. Then the more modern may be added to theſe, as *Vieta*, *Clavius*, and our *Herigon*. In like manner among the Opticians, ſhould be choſen *Vitellio*, *Kepler*, *Aquilonius*, and *Dom. de Villes*. Among Arithmeticians after *Diaphantus*, the beſt are *Cardan*, *Tartaglea*, and your Countryman *Nepper*. For Spherical Triangles, and their Computation by Logarithms, you have *Briggs*, *Gordan*, *Pitiſcus*, *Snellius*, and our *Morinus*. For Aſtronomical Matters, after *Ptolemy* and ſome *Arabians*, all thoſe ſhould be procured who have compoſed Tables, as *Alphonſus*, *John Regiomontanus*, *Kepler*, and our *Duretus*. And to be brief, for Fortification and Muſick, eight or ten Authors might be ſelected, who have moſt excell'd in the Practice. In like manner for Mechanical Affairs, the Forces of Motion, Machines, and Water-works; ſo that ten or twelve Authors being peruſed, thoſe that are curious in ſuch Matters may be eaſily ſatisfied. And if twelve Men of good Underſtanding, having a friendly Correſpondence with each other, would take this Affair upon them, ſo that each of them might compoſe a Treatiſe upon a Science, in one convenient and perſpicuous Volume; ſuch Things being ſupplied as are wanting in others, and unneceſſary Things being pared away; without doubt we might have in twelve Volumes, in a ſhort and nervous Manner, all that could be deſired in this Matter. And in my Opinion, whatever belongs to Mathematicks, either pure or mixt, might be comprehended in thoſe twelve Volumes. So one might deliver all the neater Part of Philoſophy in three Books, and all the liberal Arts or mechanic Arts in three more; ſo

Conſider'd, Oct. 1639, by Merſennus, ibid. p. 135.

ſo that Learning might be attained at a ſmall Price. Now as to Mathematical Inſtruments, it would be of little Conſequence to have an Apparatus of all that have hitherto been invented. It would be better to have four or five which are beſt in their Kind, and which ſhall be judged to be more convenient than any other.

Anſwer'd by Dr. Pell, ibid. p. 137.

3. If I underſtand you right, learned Sir, you approve of all I have ſaid, only you think I require a greater Apparatus than is neceſſary. You are of Opinion, that not all the Mathematical Books and Inſtruments ſhould be collected, which I contend for, but only the beſt ſhould be ſelected. I ſhould not oppoſe this your Opinion, if it was agreed on by all, in ſuch a Multitude of Authors which was the beſt, which ſhould have the Preference before all others; and if I had this only in view, that ſomething of Labour and Expence might be ſaved to the Studious in theſe Arts. But ſince I made it my Wiſh, to have the moſt perfect Foundation for the whole Ambit of the Mathematicks, I ought to delineate no other Scheme, than what might completely anſwer ſuch a Deſign. The principal Part of ſuch a Deſign, as I perſuade myſelf, muſt conſiſt in ſuch an univerſal Library as I have deſcribed. I ought not to deſpiſe any Attempt, much leſs to condemn any one unheard, that ſhall throw in his Mite, and endeavour to promote this Undertaking. And if I may give my Judgment, the moſt trivial Writing or Mathematical Inſtrument ſhould be preſerved, one Copy at leaſt even for its Errors, in ſome certain acceſſible Place of every Country. For we ſee many Things that have been ingeniouſly invented, in the ruder Inſtruments of former Ages, which are now not only worthy of obſerving, but even of being imitated: As ſome Writers of a lower Claſs may give very good Hints, and aſſiſt the Invention of thoſe of a happier Genius; for we can often point at an Excellence which we cannot arrive at ourſelves. We ſee many Lemmata that have been well demonſtrated by this Kind of Writers; yet becauſe of ſome one fundamental Fallacy, their whole Superſtructure has come to the Ground. If you think many are to be rejected, as well for their Trifling and Verboſity, as for Error and falſe Concluſions; you ſhould conſider how different are the Notions and Taſte of Mankind, nor ſhould you eſtimate others by your own Sagacity. For there are ſome who can underſtand nothing, unleſs repeated to them an hundred Times, and that almoſt in the ſame Words; thoſe Tautologies therefore are adapted to this Sort of People. And becauſe we muſt always begin from Things more known, but the ſame Things are not more known to all; we muſt make very different Beginnings: So that you can hardly find a Learner, but who may be aſſiſted by a rude Inſtrument or unpoliſhed Author, and therefore he that undertakes the Office of a Mathematical Monitor ſhould not be ignorant even of theſe. So that the complete Collection of Books before-mention'd ſeems to me to be quite neceſſary.

Now the more I am diſpleaſed with thoſe minute Mathematicians, the more I ſhould wiſh for a Library of this Kind, as being the only Method of curing that licentious Itch of Scribbling. For thoſe prating Pretenders, ever trifling in a childiſh Manner, while they would ſeem to accommodate themſelves to the Capacity of Youth; may fee that there are already more than enough, who have compiled Rudiments of this Kind. And they who fondly aim to advance the Mathematical Sciences by an Infinity of new Difcoveries, when they fee fo many empty Paradoxes which have been condemned and laughed at by the Publick, may take Warning by the Mifcarriages of others. But efpecially the Plagiaries, thofe Pefts of all good Literature, will not have the Impudence to vend as their own any old Books, or any Part of them, which perhaps have not been printed more than once. On the other hand, Men of Candour and Ingenuity, able to deliver their Thoughts in a handfome Manner, when they fee fo many have gone before them, on almoft every Subject, will be caution'd not to produce any Thing to the Publick but what is new and their own. Now what has been treated of already may be eafily known, either by confulting fuch an ample Library, or if they will not be at this Trouble, they may be inform'd by the Library-keeper himfelf, to whofe Cuftody I have committed it. And thefe are the Reafons in general, which I cannot retract, why I fhould prefer fuch an Univerfal Library as is above defcribed.

4. I had no fooner read your Letter, learned Sir, but I became wholly yours, and was ready to fubfcribe to your Opinion, which I intirely approve: And likewife an unufual Ardour of Mind hurried me on; fo that I would recommend this Undertaking of yours, great as it is, to the great Ones of the World, if I could have free Accefs to them. But where is the King, that will make a Beginning? For I cannot but call it a truly Royal Defign.

To the Satisfaction of Merfennus, *Dec.* 10, 1639. *Ibid. p.* 143.

5. I infpected the Mathematical Idea only by the bye, and now only remember, that I found nothing in it from which I fhould diffent; and I much approved that, firft, an Inventory of the whole Mathematical Furniture was there exhibited, and then the Self-fufficient Mathematician was defcribed, as containing every Thing in himfelf. Almoft in the fame Senfe I am ufed to diftinguifh two Things in the Mathematicks, the Hiftory and the Science. By the Hiftory, I mean all that is already found, and is committed to Books. And by the Science, the Skill of refolving all Queftions, and therefore of finding by one's own Induftry whatever can be found by human Ingenuity in that Science. He that poffeffes this Faculty does not much want Foreign Affiftance, and therefore may very properly be call'd felf-fufficient. Now it is much to be wifh'd, that this Mathematical Hiftory, which lies difperfed in many Volumes, and is not yet intire and complete, were to be all collected into one Book. Nor for this Purpofe would there be any Occafion to be at the Charge of feeking or purchafing Books. For fince Authors tranfcribe many Things from one another, nothing is extant any where which may not be fomewhere found in any Library that is but moderately furnifh'd. Nor is Diligence in collecting all Things fo neceffary, as Judgment to reject what is fuperfluous, and Knowledge to fupply fuch Things as are not yet found out. Now if fuch a Book were at hand, from thence any one might learn the whole Mathematical Hiftory, and a good Part of the Science alfo. But if any one fhould defire to have the whole that belongs to the Practice, as Inftruments, Machines, Engines, &c. if

The Judgment and Approbation of Des Cartes, *Feb.* 1640. *Ibid. p.* 144.

he was a King, and had the Wealth of the whole World at Command, he could not ſupply the neceſſary Expences. Neither indeed is there any Occaſion for them. It is enough if he can deſcribe them all, and either knows how to make ſuch as are wanting himſelf, or can ſet Artificers to work upon them.

Some of Euclid's *Propoſitions, demonſtrated independently from the reſt, by Mr.* Aſh. N. 162. *p.* 672. 32. 1. *E.*

II. THE Propoſitions which I ſhall endeavour to demonſtrate independently from all others, ſhall be theſe; the 32d and 47th of the *Firſt Book*; moſt of the *Second* and *Fifth Books*; the 1ſt and 16th of the *Sixth*; with their Corollaries. In order to demonſtrate the 32d; I ſuppoſe it known what is meant by an Angle, Triangle, Circle, External Angle, Parallels, and that the Meaſure of an Angle is the Arch of a Circle intercepted between its Sides; that a Right Angle is meaſur'd by a Quadrant, and two Right Angles by a Semi-
Fig. 1. circle. I ſay then, that in the Triangle A B C, the External Angle B C E is equal to the two oppoſite Internal ones A B C, B A C; for let a Circle be drawn, C being the Center, and B C the Radius; and let C D be drawn parallel to A B, thoſe two Lines being always equidiſtant, will both have the ſame Inclination to any third Line falling upon them; that is, (by the Definition of Angle) they will make Equal Angles with it: For if any Part of C D (for Inſtance) did incline more to B C than to A B, upon that very Account they would not be parallel; it follows therefore that the Angles A B C, B C D, are equal: Alſo B A C = D C E, becauſe A E falls upon two Parallels; but the External Angle B C E = B C D + D C E, which was before prov'd to be equal to A B C, B A C. *Q. E. D.* Hence may be inferr'd as a Corollary, That the three Angles of every Triangle are equal to two Right ones; for the Angles A C B + B C E, are meaſur'd by a Semicircle, and therefore equal to two Right Angles: Corollaries alſo from
20, 22, 31. 3. E. hence are the 20th, 22d, and 31ſt of the *Third* Book, which contain the Properties of Circles; whoſe Deduction from hence is moſt natural and obvious.

47. 1. E. In order to demonſtrate the 47th, I ſuppoſe the Meaning of the Terms made uſe of, to be known; and that two Angles or Superficies are equal, when one being put on the other, it neither exceeds, nor is exceeded. This being allow'd, I ſay, the Sides about the Right Angle are either equal or unequal; if equal, let all the Squares be deſcrib'd; the whole Figure exceeds
Fig. 2. the Square of the Hypotenuſe B C, by the two Triangles M, V, and exceeds alſo the Squares of the other two Sides, A B, A C, by the two Triangles, A B C, and S; which Exceſſes are equal, for M is equal to A B C, the two Sides about the Right Angle, being two Sides of a Square upon A B, by Suppoſition equal to A C, and the third Side equal to B C; therefore the whole Triangles are equal. After the ſame manner, S and V are proved to be equal; therefore the Square of C B is equal to the Squares of the two other Sides. *Q. E. D.*

But if the Sides be unequal, let the Squares be deſcribed, and the Paral-
Fig. 3. lelogram L Q compleated; the whole Figure exceeds the Square upon B C, by three Triangles, X, R, Z; and exceeds alſo the Square L A, A D, by the Triangle A B C, and the Parallelogram P Q: Which Exceſſes, I ſay, are equal;

equal; for Z is equal to A B C, the Side O C = B C, C D = A C, the Angle D = A, and O C D = B C A; which is manifeſt, by taking the common Angle A C O, out of the two Right Angles B C O, A C D; therefore by Superimpoſition the whole Triangles are equal. In like manner X is proved equal to A B C, alſo R; and the Parallelogram P Q to be double of the Triangle A B C: Thus the Exceſſes being proved equal, the Remainders alſo will be equal; *viz.* the Square of B C to the Square of A B, A C. *Q. E. D.* Manifeſt Corollaries from hence are the 35th and 36th of the *Third* 35, 36, 3. E.
Book; alſo the 12th and 13th of the *Second.*

The firſt Ten Propoſitions of the *Second* Book are evidently demonſtrated, 12, 13. 2. E.
only by ſubſtituting Species or Letters inſtead of Lines, and multiplying 1, 2, &c. 2 E.
them according to the Tenor of the Propoſition: Thus, to inſtance in one or two, Call the whole Line A, and its Parts B and C, therefore A = B + C, and conſequently A A = B B + C C + 2 B C, which is the very Senſe of *Fig.* 4.
the *Fourth* of the *Second* Book. Thus alſo, Let a Line be cut into equal Parts F, F, and let another Line, S, be added thereto; 'tis manifeſt, that 4 F F 4. 2. E. *Fig.* 5.
+ 4 S F + 2 S S = 2 F F + 2 F F + 2 S S + 4 S F; which is the *Tenth* Propoſition of the ſame Book. 10. 2. E.

Almoſt the whole Doctrine of Proportionals, *viz.* Permutation, Inverſion, Converſion, Compoſition, Diviſion of Ratio's, and Proportion *ex æquo*; and conſequently the moſt uſeful Propoſitions of the *Fifth* Book, are clearly demonſtrated by one Definition, and that is of Similar or Like Parts, which are ſaid to be ſuch as are after the ſame manner, or equally contain'd in their Wholes: Thus the Antecedents A and C, are either equal to their Conſequents, or greater, or leſs; if equal, the thing is manifeſt; if leſs, then (by *Fig.* 6.
the Definition of Proportionals) A and C are Like Parts of B and E; therefore what Proportions the whole, B and E, have to one another, the ſame will A and C have; which is Permutation: Likewiſe E : C : : B : A, which is Inverſion. So alſo if from Proportionals you take Like Parts, the Remainders will be proportional; whence Converſion and Diviſion are demonſtrated: And if to Proportionals you add Like Parts, the Wholes will ſtill be proportional; which is Compoſition, *&c.* If the Antecedents be greater than the Conſequents, the Conſequents will be Like Parts of them, and the Demonſtration exactly the ſame with the former.

The firſt of the *Sixth* Book is prov'd, by conſidering the Generations of 1. 6. E.
the Parallelograms, which are produced by drawing or multiplying the Perpendicular upon the Baſis; that is, taking it ſo often as there be Parts and Diviſions in the Baſe: Therefore the ſame Proportion that R X ſingle, hath to *Fig.* 7.
N X ſingle, the ſame hath R X multiplied by X Z, that is, repeated a certain number of times, to N X multiplied by X Z, that is, repeated the ſame number of times; which is as much as to ſay, R X : N X : : Paral. R Z : Paral. N Z. Now that this Propoſition is alſo true in Oblique-angled Parallelograms, is proved, becauſe they are equal to the Rectangled ones upon the ſame Baſis, and between the ſame Parallels, as does thus independently appear; the Triangles R Q X, and M P Z, are equal; for R X = M Z, Q X = P Z, R M *Fig.* 7.
= Q P; therefore adding to both M Q; R Q = M P; if therefore from theſe

equal Triangles you take what is Common, *viz*. M L Q, the Remainders will be equal, R X L M = Q L Z P; to both which add X L Z, and the whole Parallelograms will be equal, R Z = Q Z. *Q. E. D.* That Triangles alſo having a common Baſis, are in the Proportion of their Altitudes, does hence follow; becauſe they are the Halves of Parallelograms upon the ſame Baſis. This alſo is true, and the Demonſtration exactly the ſame, in Priſms, Pyramids, Cylinders, and Cones, having the ſame Baſis.

16. 6. E. *Fig.* 6. To prove the 16th of the *Sixth*, I ſuppoſe the 4 Lines, A, B, C, E, to be proportional, that is, granting A and C to be the leſſer Terms, the ſame way that A is contained in B, ſo is C in E; and that D is the Denominator of the Ratio; 'twill follow then, that B is made up of A, multiplied by D, and E of C, multiplied by D; ſo that A D = B, and C D = E; draw therefore the Extremes upon one another, that is, A upon C D, and the Means, that is, C upon A D; the Factors being the ſame, I ſay, the Products A C D, and C A D, are the ſame, and conſequently equal. *Q. E. D.*

The Problem propoſed by M. *Comier*, (with Oſtentation enough) as if it contain'd ſomething New, though in reality it be nothing but the Old Buſineſs of doubling the Cube, a little diſguiſed, is eaſily ſolved Algebraically, as follows,

Fig. 8.

8. 6. E.	1	$a : 2x :: x : \frac{2x^2}{a} = p$
47. 1. E.	2	$aa + 2ax - \frac{2x^3}{a} = \frac{4x^4}{aa}$
2 × by aa	3	$a^4 + 2a^3x = 4x^4 + 2x^3a$
$3 \div \overline{a + 2x}$	4	$a^3 = 2x^3$; that is, The Cube upon x is half the Cube upon a.

The Squaring of the Hyperbola, by the Lord Viſcount Brounker. N. 34. *p*. 645. *Fig.* 9.

III. Let A B be one Aſymptote of the Hyperbola E d C; and let A E, and B C, be parallel to the other: Let alſo A E be to B C, as 2 to 1; and let the Parallelogram A B D E be equal to 1.

Suppoſing the Reader knows, that E A, $\alpha\zeta$, K H, $\beta\eta$, $d\theta$, $\gamma\kappa$, $\delta\lambda$, $\epsilon\mu$, C B, *&c.* are in an Harmonick Series, or a Series *Reciproca-Primanorum, ſeu Arithmetice Proportionalium*, (otherwiſe he is referr'd for Satisfaction to *Arithm. Infinitor. Walliſii*, Prop. 87, 88, 89, *&c.*) I ſay,

$$ABCdEA = \frac{1}{1\times2} + \frac{1}{3\times4} + \frac{1}{5\times6} + \frac{1}{7\times8} + \frac{1}{9\times10}, \text{ \&c.}$$

$$EdDE = \frac{1}{2\times3} + \frac{1}{4\times5} + \frac{1}{6\times7} + \frac{1}{8\times9} + \frac{1}{10\times11}, \text{ \&c.}$$

$$EdCyB = \frac{1}{2\times3\times4} + \frac{1}{4\times5\times6} + \frac{1}{6\times7\times8} + \frac{1}{8\times9\times10}, \text{ \&c.}$$

For (in *Fig.* 10. and 11.) the *Parallelogram*, And (in *Fig.* 12.) the *Triangle*, *Fig.* 10, 11, 12.

	$CA = \frac{1}{1\times2}$	$EdC = \frac{1}{2\times3\times4} = \frac{\square dD - \square Fd}{2}$
$dD = \frac{1}{2\times3}$	$dF = \frac{1}{3\times4}$	$Ebd = \frac{1}{4\times5\times6} = \frac{\square br - \square bn}{2}$
$br = \frac{1}{4\times5}$	$bn = \frac{1}{5\times6}$	$dfC = \frac{1}{6\times7\times8} = \frac{\square fG - \square fk}{2}$
$fG = \frac{1}{6\times7}$	$fk = \frac{1}{7\times8}$	$Eab = \frac{1}{8\times9\times10} = \frac{\square aq - \square ap}{2}$
$aq = \frac{1}{8\times9}$	$ap = \frac{1}{9\times10}$	$bcd = \frac{1}{10\times11\times12} = \frac{\square cs - \square cm}{2}$
$cs = \frac{1}{10\times11}$	$cm = \frac{1}{11\times12}$	$def = \frac{1}{12\times13\times14} = \frac{\square et - \square el}{2}$
$et = \frac{1}{12\times13}$	$el = \frac{1}{13\times14}$	$fgC = \frac{1}{14\times15\times16} = \frac{\square gu - \square gh}{2}$
$gu = \frac{1}{14\times15}$	$gh = \frac{1}{15\times16}$	*&c.*
&c.	*&c.*	

Note,

Note, $\frac{1}{2}CA = dD + dF$
$\frac{1}{2}dD = br + bn$
$\frac{1}{2}dF = fG + fk$
$\frac{1}{2}br = aq + ap$
$\frac{1}{2}bn = cs + cm$
$\frac{1}{2}fG = et + el$
$\frac{1}{2}fk = gu + gh$, &c.

And that therefore in the firſt Series half the firſt Term is greater than the Sum of the two next, and half this Sum of the Second and Third greater than the Sum of the four next; and half the Sum of thoſe Four greater than the Sum of the next Eight, *&c. in infinitum.* For $\frac{1}{2}\,dD = br + bn$; but $bn > fG$, therefore $\frac{1}{2}\,dD > br + fG$, *&c.* And in the Second Series, half the ſecond Term is leſs than the Sum of the two next, and half this Sum leſs than the Sum of the four next, *&c. in infinitum.*

That the firſt Series are the even Terms, *viz.* the 2d, 4th, 6th, 8th, 10th, *&c.* and the ſecond the odd, *viz.* the 1ſt, 3d, 5th, 7th, 9th, *&c.* of the following Series, *&c.*

$$\frac{1}{1\times2},\ \frac{1}{2\times3},\ \frac{1}{3\times4},\ \frac{1}{4\times5},\ \frac{1}{5\times6},\ \frac{1}{6\times7},\ \text{\&c. } \textit{in infinitum.} = 9.$$

Whereof a, being put for the Number of Terms taken at Pleaſure, $\frac{1}{a^2+a}$ is the laſt, $\frac{a}{a+1}$ is the Sum of all thoſe Terms from the Beginning, and $\frac{1}{a+1}$ the Sum of the reſt to the End.

That $\frac{1}{4}$ of the firſt Term in the third Series is leſs than the Sum of the two next, and $\frac{1}{4}$ of this Sum leſs than the Sum of the four next, and $\frac{1}{4}$ of this laſt Sum leſs than the next eight, I thus demonſtrate:

Let a be equal to the third or laſt Number of any Term of the firſt Column, *viz.* of Diviſors;

$$\frac{1}{a\times\overline{a-1}\times\overline{a-2}} = \frac{1}{a^3-3a^2+2a} = \frac{16a^3-48a^2}{16a^6-96a^5+232a^4-288a^3+184a^2-48a}$$

$$\frac{+56a-24}{} = \text{A.}$$

$$\left.\begin{array}{l}\dfrac{1}{2a\times\overline{2a-1}\times\overline{2a-2}} = \dfrac{1}{8a^3-12a^2+4a}\\[2ex] \dfrac{1}{\overline{2a-2}\times\overline{2a-3}\times\overline{2a-4}} = \dfrac{1}{8a^3-36a^2+52a-24}\end{array}\right\} =$$

$$\frac{16a^3 - 48a^2 + 56a - 24}{64a^6 - 384a^5 + 880a^4 - 960a^3 + 496a^2 - 96a} = B.$$

$$\frac{64a^6 - 384a^5 + 928a^4 - 1152a^3 + 736a^2 - 192a}{64a^6 - 384a^5 + 880a^4 - 960a^3 + 496a^2 - 96a} \times \tfrac{1}{4}A = B.$$

And $48a^4 - 192a^3 + 240a^2 - 96a$ = Excess of the Numerator above the Denominator.

$$\left.\begin{array}{ll} \text{But the Affirmat.} & > \text{the Negat.} \\ \text{That is, } 48a^4 + 240a^2 & > 192a^3 + 96a \\ \text{Because } a^4 + 5a^2 & > 4a^3 + 2a \\ a^3 + 5a & > 4a^2 + 2 \\ \text{Therefore } B & > \tfrac{1}{4}A. \end{array}\right\} \text{if } a > 2.$$

Therefore one fourth of any Number of A, or Terms, is less than their so many respective B; that is, than twice so many of the next Terms. *Q. E. D.*

By any one of which three Series it is not hard to calculate, as near as you please, these and the other like Hyperbolic Spaces, whatever be the rational Proportion of A E to B C; as for Example, When A E is to B C, as 5 is to 4. (Whereof the Calculation follows, after that where the Proportion is, as 2 to 1; and both by the *Third* Series.)

First then, when (in *Fig.* 9.) A E : B C :: 2 : 1. *Fig.* 9.

2× 3× 4) 1.	(0. 0416666666]	0. 0416666666
4× 5× 6) 1.	(0. 0083333333	0. 0113095237
6× 7× 8) 1.	(0. 0029761904	
8× 9×10) 1.	(0. 0013888888	0. 0029019589
10×11×12) 1.	(0. 0007575757	
12×13×14) 1.	(0. 0004578754	
14×15×16) 1.	(0. 0002976190	
16×17×18) 1.	(0. 0002042484	0. 0007306482
18×19×20) 1.	(0. 0001461988	
20×21×22) 1.	(0. 0001082251	
22×23×24) 1.	(0. 0000823452	
24×25×26) 1.	(0. 0000641026	
26×27×28) 1.	(0. 0000508758	
28×29×30) 1.	(0. 0000410509	
30×31×32) 1.	(0. 0000336021	

32×33

32×33×34)	1.	(0. 0000278520	
34×35×36)	1.	(0. 0000233426	
36×37×38)	1.	(0. 0000197566	
38×39×40)	1.	(0. 0000168691	
40×41×42)	1.	(0. 0000145180	
42×43×44)	1.	(0. 0000125843	
44×45×46)	1.	(0. 0000109793	
46×47×48)	1.	(0. 0000096361	
48×49×50)	1.	(0. 0000085034	> 0. 0001829939
50×51×52)	1.	(0. 0000075415	
52×53×54)	1.	(0. 0000067193	
54×55×56)	1.	(0. 0000060125	
56×57×58)	1.	(0. 0000054014	
58×59×60)	1.	(0. 0000048704	
60×61×62)	1.	(0. 0000044068	
62×63×64)	1.	(0. 0000040002	

0. 0416666666
0. 0110095237
0. 0029019589
0. 0007306482
3)0. 0001829939 (0. 0000609980

0. 05679179
+0. 00006100

0. 05685279 < E d C y

But 0. 0007306482
0. 0001829939 } ∺
0. 0000458315

Therefore 0. 05679179
+0. 00004583
+0. 00001528

0. 05685290 > E d C y.

For it has been demonſtrated, That $\frac{1}{4}$ of any Term in the laſt Column is leſs than the Term next after it; and therefore that $\frac{1}{3}$ of the laſt Term, at which you ſtop, is leſs than the remaining Terms; and that the Total of theſe is leſs than $\frac{1}{4}$ of a third Proportional to the two laſt.

And therefore A B C *y* E being = 0· 75 ———— 0. 57
and E *d* C *y* > 0. 05685279 —— and < 0. 05685290

And A B C *d* E is < 0. 69314720 —— and > 0. 69314709

But when A E : B C :: 5 : 4, or as E A to K H; then will the Space A B C E, or now, the Space A H K E (A H = $\frac{1}{4}$ A B) be found as follows. *Fig.* 9.

8x 9x10) 1. (0. 0013888888] 0. 0013888888

16x17x18) 1. (0. 0002042484 }
18x19x20) 1. (0. 0001461988 } 0. 0003504472

32x33x34) 1. (0. 0000278520 }
34x35x36) 1. (0. 0000233426 }
36x37x38) 1. (0. 0000197566 } 0. 0000878204
38x39x40) 1. (0. 0000168691 }

0. 0013888888
0. 0003504472
3) 0. 0000878204 (0. 0000292735

0. 0018271564
+0. 0000292735

0. 0018564299 < E *a b*

But 0. 0003504472 }
0. 0000878204 } ÷
0. 00002200737 }

Therefore 0. 0018271564
+0. 0000220074
+0. 0000073358

0. 0018564296 > E *a b*

Therefore E M *b* (*Fig.* 12.) being = 9. 025 ———— 0. 025
E *a b* > 0. 0018564292 and < 0. 0018564996

E M *b a* (*Fig.* 12.) or E K M (*Fig.* 9.) > 0. 02685643 < 0. 02685650
A H K E < 0. 22314356 < 0. 22314349

Therefore

Therefore 3 ABC*d*E = 2. 37944154
and AHKE = 0. 2231435

ABC*d*E (when AE : BC :: 10 : 1.) = 2. 3025850

Therefore, The Logar. of 10, is to the Log. of 2, as 2. 302585, to 0. 693147.

The Quadrature of a Circle, by M. Leibnitz, *Phil. Coll.* N. 7. *p.* 204.

IV. The Quadrature of the Circle, or the turning it into an equal Share, or any other Right-lin'd Figure, (which depends upon the Ratio of the Circle to the Square of its Diameter, or of the Circumference to its Diameter) may be underſtood to be fourfold, to wit, either by Calculation, or by Linear Conſtruction; and each of them again may be either perfectly exact, or elſe almoſt, or pretty near. The Quadrature by accurate Calculation, I call the Analytical; That which is done by accurate Conſtruction, I call the Geometrical: That which is done by Calculation pretty near, I call the Approach; that which is by Conſtruction pretty near, I call Mechanical.

The Approaches have been furtheſt carried on by *Ludolph van Ceulen* : *Vieta*, *Hugenius*, and others, have given ſeveral Mechanical. The accurate Geometrical Conſtruction may be had, by which not only an entire Circle may be meaſured, but any Section or Arch of it alſo; which is by an exact and ordinate Motion, but ſuch notwithſtanding as ſuits with Tranſcendental Curves, which erroneouſly are accounted Mechanical, though in truth they are as Geometrical as thoſe which are commonly ſo eſteemed; though they are not Algebraical, nor can be reduced to Equations Algebraical, or of certain Degrees; they having Degrees proper to themſelves, which though they be not Algebraical, are yet nevertheleſs Analytical.

The Analytical Quadrature, or that which is made by accurate Calculation, may be again ſubdivided into three Kinds; namely, into the Analytical Tranſcendent, the Algebraical, and the Arithmetical. The Analytical Tranſcendent is to be obtained, amongſt others, by Equations of Degrees indefinite, hitherto conſidered by none: As if $X^x + X$ be equal to 30, and X be ſought, it will be found to be 3; becauſe $3^3 + 3$, is $27 + 3$, or 30.

The Algebraical is done by Vulgar Numbers, though irrationally, or by the Roots of common Equations; which for the general Quadrature of the Circle, or its Sectors, is indeed impoſſible. Now there remains the Arithmetical Quadrature, which is performed by certain Series exhibiting the Quantity of the Circle exact by a Progreſſion of Terms (firſt) Rational, ſuch as I ſhall here propound.

I have found therefore, that if the Square of the Diameter be put 1, the Area of the Circle will be $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17}$, &c. to wit, the entire Square of the Diameter being diminiſhed (that it may not be too big) by a third Part; and again (becauſe hereby too much is taken away) being augmented by one fifth; and again (becauſe by this laſt too much is added) diminiſhed by one ſeventh; and ſo onward, continually.

And

And the firſt Quantity will be too great, *viz.* 1. but the Error will be leſs than ———————————————— $\frac{1}{3}$

The next too little, *viz.* $1 - \frac{1}{3}$, but the Error will be leſs than $\frac{1}{5}$

The 3d, too much, *viz.* $1 - \frac{1}{3} + \frac{1}{5}$, but, &c. - - - - - - - $\frac{1}{7}$

The 4th too little, *viz.* $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$, but, &c. - - - - $\frac{1}{9}$ &c.

&c.

The whole Series contains all the Approaches together, or the Values both greater than they ought to be, and leſs than they ought to be.

So that by continuing the Series, the Errors may be made leſs than any Fraction given, and conſequently leſs than any aſſignable Quantity. Whence it follows, that the whole Series muſt give the true Value. And though the Sum of the whole Series cannot be expreſſed by one Number, and that the Series be infinitely continued; yet becauſe it conſiſteth of one regular Method of Progreſſion, the whole may ſufficiently enough be conceived by the Mind. And if *Van Ceulen* could have given a Rule by which his Numbers 314159, &c. could have been continued *in infinitum*, he would have given us the Arithmetical Quadrature exact in whole Numbers, which we have here done in Fractions.

There are ſeveral things relating to this Quadrature which might be taken notice of, eſpecially one, *viz.* That the Terms of this our Series, $\frac{1}{1}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, &c. are of Harmonical Progreſſion, or in a continued Harmonical Progreſſion; as will be evident to any one that ſhall examine them. And a Series made by ſkipping, as $\frac{1}{1}$, $\frac{1}{5}$, $\frac{1}{9}$, $\frac{1}{13}$, $\frac{1}{17}$, &c. is alſo of Harmonical Progreſſion. And $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{11}$, $\frac{1}{15}$, $\frac{1}{19}$, &c. is alſo a Series of Harmonical Proportionals. Wherefore ſince the Circle is $\frac{1}{1} + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \frac{1}{17}$, &c. $- \frac{1}{3} - \frac{1}{7} - \frac{1}{11} - \frac{1}{15} - \frac{1}{19}$, &c. by ſubſtracting the latter partial Series from the former partial Series, the Circle will be the Difference of two Series in Harmonical Progreſſion. And becauſe the Sum of any Number of Terms in Harmonical Progreſſion, how many ſoever, may by ſome Compendium be obtained; hence Compendious Approaches (if after *Van Ceulen* there be any need of them) may be deduced; if one would in this our Series take out the Terms affected with the Sign —, by adding the two next into one, +

 $\frac{1}{1}$

$\frac{1}{1} - \frac{1}{3}$, and $+ \frac{1}{5} - \frac{1}{7}$, and $+ \frac{1}{9} - \frac{1}{11}$, and $+ \frac{1}{13} - \frac{1}{15}$, and $+ \frac{1}{17} - \frac{1}{19}$, and so onward, he will have a new Series for the Magnitude of the Circle, namely, $\frac{2}{3}$ (that is, $\frac{1}{1} - \frac{1}{3}$) $+ \frac{2}{35}$ (that is, $\frac{1}{5} - \frac{1}{7}$) $+ \frac{2}{99}$ (that is $\frac{1}{9} - \frac{1}{11}$) &c. Wherefore

- - - - - The Square inscrib'd being $\frac{1}{4}$

The Area of the Circle shall be $\frac{1}{3} + \frac{1}{35} + \frac{1}{99} + \frac{1}{195} + \frac{1}{323}$, &c.

But the Numbers 3, 35, 99, 195, 323, &c. by Skipping are taken out of the Series of Square Numbers, 4, 9, 16, 25, &c. diminished by an Unite, and so made the Series 3, 8, 15, 24, &c. out of the Members of which Series, every fourth after the first, is a Number of this our Series. But I have found (which is worth nothing) the Sum of this infinite Series, $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} +$ &c. to be $\frac{3}{4}$. Nay, and by culling out by single Skipping, as $\frac{1}{3} + \frac{1}{15} + \frac{1}{35}$, &c. the Sum of this infinite Series maketh $\frac{2}{4}$ or $\frac{1}{2}$. But if out of this again another Progression be culled by single Skipping, as $\frac{1}{5} + \frac{1}{35} + \frac{1}{99}$, &c. the Sum of that infinite Series shall be the Semicircle, the Square of the Diameter being 1. Now because by the same Means the Arithmetical Quadrature of the *Hyperbola* is obtain'd, I thought it not amiss to represent to View the whole Harmony.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	&c.	
1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400	&c.	
0	3	8	15	24	35	48	63	80	99	120	143	168	195	224	255	288	323	360	399	&c.	
.	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{1}{15}$	$\frac{1}{24}$	$\frac{1}{35}$	$\frac{1}{48}$	$\frac{1}{63}$	$\frac{1}{80}$	$\frac{1}{99}$	$\frac{1}{120}$	$\frac{1}{143}$	$\frac{1}{168}$	$\frac{1}{195}$	$\frac{1}{224}$	$\frac{1}{255}$	$\frac{1}{288}$	$\frac{1}{323}$	$\frac{1}{360}$	$\frac{1}{399}$	&c.	$= \frac{3}{4}$
.	$\frac{1}{3}$	.	$\frac{1}{15}$	.	$\frac{1}{35}$	.	$\frac{1}{63}$	.	$\frac{1}{99}$	.	$\frac{1}{143}$	.	$\frac{1}{195}$	.	$\frac{1}{255}$	.	$\frac{1}{323}$	.	$\frac{1}{399}$	&c.	$= \frac{2}{4}$
.	.	$\frac{1}{8}$	.	$\frac{1}{24}$	.	$\frac{1}{48}$	.	$\frac{1}{80}$	.	$\frac{1}{120}$	.	$\frac{1}{168}$	.	$\frac{1}{224}$	.	$\frac{1}{288}$	.	$\frac{1}{360}$	.	&c.	$= \frac{1}{4}$
.	$\frac{1}{3}$	.	.	.	$\frac{1}{35}$	.	.	.	$\frac{1}{99}$	.	.	.	$\frac{1}{195}$	.	.	.	$\frac{1}{323}$	.	.	&c.	= to
.	.	$\frac{1}{8}$	.	.	.	$\frac{1}{48}$	.	.	.	$\frac{1}{120}$	.	.	.	$\frac{1}{224}$	.	.	.	$\frac{1}{360}$	.	&c.	= to

the Circle A B C D, whose Inscribed Square is }
the Hyperbola CBEHC, whose Power, A B C D, is } $\frac{1}{4}$

Fig. 13. To the Asymptotes A F, A E, at Right Angles to each other, let there be described the Curve Line of an Hyperbola G C H, whose Vertex is C; and A B C D, the Power or Square, to which every Rectangle made of the Ordinate,

nate, as E H, and the intercepted Part, A E, is always equal. About this Square let a Circle be drawn, and let the Hyperbola be continued from C to H, ſo that A E be double to A B. Then putting A E to be 1, A B ſhall be $\frac{1}{2}$, and its Square, ABCD, ſhall be $\frac{1}{4}$, and the Circle, (whoſe Power A B C D is inſcrib'd) ſhall be $\frac{1}{3} + \frac{1}{35} + \frac{1}{99}$, *&c.* but the Portion of the Hyperbola CBEHC (whoſe Power inſcrib'd is the ſame Square $\frac{1}{4}$) which repreſents the Logarithm of the Ratio of A E to A B, (or of 2 to 1) ſhall be $\frac{1}{8} + \frac{1}{48} + \frac{1}{120}$, *&c.*

Tangents to all Geometrical Curves; by Renatus Fran. Sluſius. n. 90. *p.* 513. Jan. An. 1673. *Fig.* 14.

V. 1. Let any Curve D Q be given, all the Points of which may be referred to any given Right Line E A B, by the Right Line D A; whether E A B be the Diameter, or any other Line, or any other Lines are given at the ſame time, any of which, or their Powers, enter the Equation of the Curve, it matters little.

In the Analytical Equation, for the more eaſy Explanation, let D A be always called v, and B A y. And let E B, and all the other given Quantities, be denoted by Conſonants.

Then ſuppoſe the Line D C to be drawn, touching the Curve in D, and meeting E B (produced if neceſſary) in the Point C; and let C A be always called a. Now to find A C, or a, this will be the general Rule.

1. Rejecting thoſe Parts of the Equation in which y or v are not found, let all thoſe be ſet on one Side of the Equation in which is y, and thoſe on the other Side in which is v, with their Signs + or —. For Diſtinction Sake we will call this the Right Side, and that the Left.

2. On the Right Side, let there be prefixt the Exponent of the Power of v to all the Parts, or, which is the ſame thing, let the Parts be multiplied by it.

3. Let the ſame thing be done on the Left Side, that is, prefix to each Part the Exponent of the Power of y: And farther, in each of the Parts let one y be changed into a.

I ſay that the Equation ſo form'd will ſhew the Manner of drawing the Tangent at the given Point D. For when it is given, at the ſame time are given y and v, and the other Quantities which are expreſſed by Conſonants; and then a will be known alſo.

If the Rule ſhall ſeem to be any thing obſcure, it may be thus illuſtrated by ſome Examples. Let this Equation be given, $by - yy = vv$; in which let E B be b; B A, y; D A, v; and let a or A C be ſought, ſuch as D C being joined may touch the Curve D Q in D. By the Rule nothing is here to be rejected, becauſe either y or v is found in every Part of the Equation. Alſo it is ſo diſpoſed, that all the Parts in which y is found are on one Side, and all in which there is v are on the other. Therefore we need only prefix to each the Exponents of the Powers of y and v; and on the Left Side one y is to be changed into a. Therefore it will be $ba - 2ya = 2vv$. I ſay now, that this Equation will ſhew the Manner of drawing the Tangent to the Point D; that is, $a = \frac{2vv}{b - 2y} = AC$.

Thus if the Equation $qq + by - yy = vv$ were given, the Equation for the Tangent would be the ſame as the former; for we muſt reject qq as the Rule preſcribes.

Thus from the Equation $2byy - y^3 = v^3$, we have $4bya - 3yya = 3v^3$, or $a = \frac{3v^3}{4by - 3yy}$. And from the Equation $bby + zyy + y^3 = qvv$, we ſhall have $bba + 2zya + 3yya = 2qvv$, and thence $a = \frac{2qvv}{bb + 2zy + 3yy}$. And from $b^4 + by^3 - y^4 = qqvv + zv^3$, we ſhall have $3byya - 4y^3a = 2qqvv + 3zv^3$, and then $a = \frac{2qqvv + 3zv^3}{3byy - 4y^3}$.

And in all other Equations of this kind I think no Difficulty can ariſe. Perhaps there may be ſome in ſuch Equations, ſome Parts of which conſiſt of Products of y into v. As yv, yyv, y^3v^2, and ſuch like. But this Difficulty will be but a ſlight one, as will appear from theſe Examples. For let there be given $y^3 = bv^2 - yv^2$. Now there is nothing to be rejected from hence, ſince y or v is found in all the Parts or Terms.

But that they may be diſpoſed as the Rule preſcribes, the Term yv^2 is to be aſſumed twice, and placed as well on the Right Side of the Equation, in which are theſe Terms that have v, as on the Left Side, where are the Terms that have y; ſince yv^2 contains both y and v. Therefore we muſt make $y^3 + v^2y = bv^2 - yv^2$. Then changing, as before, this Equation into another $3yya + vva = 2bvv - 2yvv$, we ſhall have $a = \frac{2bvv - 2yvv}{3yy + vv}$.

For the Rule is to be ſo underſtood, that on the Left Side the Power of v is not conſidered, and therefore the Exponent of v^2 is not to be prefixed to yv^2, but only that of y. And on the contrary on the other Side, the Power of y is not to be conſidered in yvv, but only that of v, and to this its Exponent is to be prefixed. Thus if the Equation were $y^5 + by^4 = 2qqv^3 - y^2v^3$, we muſt make $y^5 + by^4 + v^3y^2 = 2qqv^3 - y^2v^3$, and we ſhould have the Equation for the Tangent $5y^4a + 4by^3a + 2v^3ya = 6q^2v^3 - 3y^2v^3$, and thence $a = \frac{6q^2v^3 - 3y^2v^3}{5y^4 + 4by^3 + 2v^3y}$.

And in theſe Examples I think I have comprehended all the Variety of Caſes that can be propoſed. But perhaps it may be of ſome uſe to apply what I have delivered in general, to ſome particular Curve. Therefore let the Curve BD be given, which has this Property, that taking any Point in
Fig. 15. it D, if BD be joined, and DE be raiſed perpendicular to it, meeting the Right

Right Line B E in E, the Right Line D E may always be equal to the given Right Line B F.

That we may have an Equation in Analytical Terms, make $DA = v$, $BA = y$, BF or $DE = q$. Therefore it will be $EA = \frac{vv}{y}$. And ſince the Square of D E is equal to the two Squares of D A and A E, we ſhall have the Equation $qq = \frac{v^4}{y^2} + v^2$; or $q^2y^2 = v^4 + y^2v^2$; which for the Tangent, as preſcribed by the Rule, is to be reform'd to $qqy^2 - v^2y^2 = v^4 + y^2v^2$, and $2qqya - 2v^2ya = 4v^4 + 2y^2v^2$, whence $a = \frac{2v^4 + y^2v^2}{q^2y - v^2y}$.

Now how ſuch Equations are to be reduced to eaſier Terms, for the Sake of the Conſtruction, will be no Myſtery to ſkilful Mathematicians. As in this Example, becauſe the Rectangle B A E is ſuppoſed equal to the Square of A D, if E A be called e, it will be $vv = ye$, and $v^4 = y^2e^2$, and $qq = ye + ee$. Therefore inſtead of them ſubſtituting their Values in the Equation, it becomes $a = \frac{2y^2e^2 + y^3e}{e^2y + ey^2 - ey^2}$, or $a = \frac{2ey + yy}{e}$, that is, $ae = 2ey + yy$, and adding ee on both Sides, $ae + ee = ee + 2ey + yy = \overline{e+y}|^2$. So that the three Quantities e, $e + y$, $a + e$, or E A, E B, E C, are in continual Proportion; whence the Conſtruction becomes very eaſy.

But whereas we ſeem to have ſuppoſed hitherto, that the Tangent is always to be drawn towards B, when it may happen from the given Quantities, that it ſhould be either parallel to A B, or ſhould be drawn the contrary Way; we muſt now determine how this Diverſity of Caſes is to be diſtinguiſhed in Equations. A Fraction therefore being made for the Value of a, as in the foregoing Examples, the Parts of the Numerator and the Denominator are to be conſidered, as alſo their Signs.

1. For in both if either all the Parts have the Sign +, or at leaſt if the Affirmative prevail over the Negative, the Tangent muſt be drawn towards B.

2. If in the Numerator the Affirmative prevail over the Negative, but are equal in the Denominator, a Line drawn through D parallel to A B will touch the Curve in D: For in this caſe a will be of an infinite Length.

3. If as well in the Denominator as in the Numerator the affirmative Parts are leſs than the Negative; then all the Signs being changed, the Tangent muſt ſtill be drawn towards B: For this caſe coincides with the firſt.

4. If in the Denominator they prevail, but in the Numerator are leſs; the Tangent muſt be drawn on the contrary Side, that is, A C muſt be taken towards E.

5. Laſtly, if in the Numerator the affirmative Parts are equal to the Negative, however they may be in the Denominator, then a will become nothing. Therefore either A D will be the Tangent, or E A, or parallel to it, which is eaſily known from the given Quantities. Now all the Variety of heſe Caſes may be ſhewn by the Equation to the Circle.

For

Fig. 16. For let there be a Semicircle whose Diameter is E B, and let D be any given Point in it, from whence let fall the perpendicular D A $= v$. Make B A $= y$, B E $= b$; the Equation will be $by - yy = vv$. And drawing the Tangent D C, it will be A C or $a = \frac{2vv}{b - 2y}$. Now if b be greater than $2y$; the Tangent muſt be drawn towards B; if equal, it is parallel to E B; if leſs, it muſt be drawn towards E. See 1, 2, 4.

Fig. 17. Again, let there be given another Semicircle inverted, whoſe Points are to be conceived as referred to a Right Line parallel to the Diameter, and equal to the ſame, as in the Scheme. The Parts being denominated as before, and making N B $= d$, the Equation will be $by - yy = dd + vv - 2dv$. Therefore A C or $a = \frac{2vv - 2dv}{b - 2y}$. Now ſince by the Example we have ſuppoſed v always to be leſs than d; if b is greater than $2y$, the Tangent muſt be drawn towards E; if equal, it will be parallel; if leſs, then changing all the Signs, it muſt be drawn towards B; as 4, 5, 3. But no Tangent would be drawn, or E B itſelf would be the Tangent, if we ſhould ſuppoſe N B to be equal to the Semidiameter, or $2d = b$ as at *n.* 5.

Fig. 18. Laſtly, let there be another Semicircle, whoſe Diameter N B is perpendicular to the Right Line B E, to which its Points are ſuppoſed to be referred. Let N B be called b, and let the other Parts be named as before. The Equation will be $yy = bv - vv$, and thence $a = \frac{bv - 2vv}{2y}$. Now if b is greater than $2v$, the Tangent muſt be drawn towards B; if leſs, towards E; if equal, D A itſelf will be the Tangent. As by 1, 4, 5.

And this, I think, is all the Variety of Caſes, which can be obſerved from the Conſideration of Equations.

The Lemmata, *whereby the preceding Method is demonſtrated, by* M. Sluſius. n. 95. *p.* 6059. n. 97. *p.* 6126. June, An. 1673.

(1) The Difference of two Dignities of the ſame Degree, applied to the Difference of the Sides, will give the ſeveral Parts of the inferior Degree of a Binomial of the Sides. Thus $\frac{y^3 - x^3}{y - x} = y^2 + yx + x^2$; as is eaſy to prove.

(2.) There are ſo many Parts ariſing from a Binomial of any Degree, as the Exponent of the Dignity next above has Units. That is, three in the Square, four in the Cube, *&c.* And this is generally known.

(3.) If the ſame Quantity is applied to two others, whoſe Ratio is given, the Quotients will be in the ſame Ratio reciprocally.

By theſe Lemmata my Method may eaſily be demonſtrated, they being here diſpoſed in ſuch an Order, as will naturally lead to the Demonſtration.

The Teſtudo Veliformis Quadrabilis *Ænigmatically propos'd by* V. V. n. 196. *p.* 584. Jan. An. 1693.

VI. 1. *A Geometrical* Æ N I G M A, *concerning the wonderful Structure of a Hemiſpherical Cupola which is quadrable. By D.* Pio Liſci, *a minute Mathematician.*

Among the venerable Monuments of ancient learned *Greece*, there is ſtill in Being, and for ever will endure, a moſt Auguſt Temple dedicated to *Divine Geometry*, of a Circular Ichnography, which is cover'd by a Cupola perfectly

perfectly Hemispherical within. But in this there are the equal Areas of four Windows, (disposed about and upon the Base of the Hemisphere) of such a Configuration, Amplitude, and of such Ingenuity, that these being taken away, the remaining curved Superficies of the Cupola, adorned with curious Mosaick Work, is capable of a true Geometrical Quadrature.

Now it is inquired, which is that quadrable Part of the curved Hemispherical Superficies, which is like a distended Nautical Sail, and by what Art or Method the Geometrical Architect attained to it; and lastly, what quadrable Geometrical Plain is it equal to?

2. A Day or two ago, just as I was going to Bed, I received (learned Sir) your Letter, which being very busy I could not answer Yesterday. Within was inclosed a little printed Paper, which, you say, you received from *Florence*, to be transmitted to me. *Solved by Dr. Wallis, ibid. p. 587.*

That Paper contains a Geometrical Ænigma, which when stript of its verbal Disguise seems to insinuate the following Problem. From the curved Superficies of a Hemisphere to cut off four equal Segments, so that what remains shall be capable of Quadrature.

And at the same Time it gives us a Hint, that something yet remains among the *Grecian* Monuments, by means of which this may be done.

I imagine this to be the Quadrature of the Lunula of *Hippocrates* of *Chius*.

For whereas *Archimedes* has demonstrated, that the curved Superficies of an Hemisphere is equal to two great Circles of the same Sphere, that is, to four Semicircles; and *Hippocrates* of *Chius* has taught us how to square a certain Lunula: If from every Quarter of this Hemispherical Cupola so much be taken away as that Lunula is deficient from a Semicircle, the Remainder will be equal to the Square that may be inscribed in the great Circle of the Sphere, on which the Hemispherical Cupola insists.

Now if besides the Ænigmatical Involution of the Problem, any Historical Fact about a Temple should be intended, I should imagine the Temple here meant is that of *Sancta Sophia* at *Constantinople*.

SCHOLIUM.] By the Quadrature of the Lunula of *Hippocrates* of *Chius*, (mentioned in the first Book of *Aristotle's* Physicks, and in the Commentaries of *Simplicius* upon that Place) the Semicircle A B D being divided into two Quadrants, A C D, B C D, if the Subtense of the Quadrantal Arch A D be adapted, bisected in H by the Radius C E, and if Center H the Semicircle A D F is described: Because of the Square of the Right Line A D being subduple of the Square of the Right Line A B, the Semicircle A D F will be subduple of the Semicircle A B D, and therefore will be equal to the Quadrant A C D. Now taking away the common Segment A D E from each, the remaining Lunula A E D F will be equal to the remaining Triangle A D C. And four such *Lunulæ* will be equal to four such Triangles, that is, to the whole Square A D B G inscribed in the Circle. *Fig. 19.*

Again, by what *Archimedes* has demonstrated, the Surface of the Sphere is equal to four of its great Circles; and therefore the curved Superficies of an an Hemiſphere is equal to four ſuch Semicircles, and a Quadrant of ſuch an Hemiſpherical Surface is equal to one Semicircle.

Fig. 20. Let the Circle A D B G be now the Baſe of a curved Hemiſpherical Surface, whoſe Pole is P, its Axis C P perpendicular to the Plain of the Baſe, and one of its Quadrants D P A, which is biſected by a Plain E P C paſſing through the Axis.

Alſo for the Convenience of Calculation, let the Radius of the Circle be call'd R, the Diameter $D = 2$ R, the Periphery P, and the Arch exhibited *a*.

And putting the Quadrantal Arch $DEA = a = \frac{1}{4}P$, the Semicircle ABD is $aR = \frac{1}{4}RP$: The Triangle $ADC = \frac{1}{2}R^2 = \frac{1}{4}RD$. And the Remainder of the Semicircle, when this Triangle is taken away, will be $\frac{1}{4}RP - RD$; an equal to which is to be taken away from D P A, a Quadrant of the curved Hemiſpherical Surface, equal to the Semicircle A B D, that the Remainder may be equal to the exhibited Triangle A D C.

Now ſince this may be done various Ways, by what we have ſhew'd long ago, *An.* 1659. (at the End of the Treatiſe of the *Cycloid* then publiſh'd, p. 122. at § 68.) and again *An.* 1670, (in the Treatiſe concerning Motion, C. V. P. 24.) concerning a Plain Figure, which is equal to any Figure in a Spherical Surface, that is terminated by any Circles greater or leſſer. It may be thus done in the ſimpleſt Manner.

Fig. 21. Since the Segments of a Spherical Surface, cut off by parallel Plains, are proportional to the Segments of the Axis; (which obtains likewiſe in the exhibited Superficies D P A of the Quadrantal Cuneus) if in the Axis C P there be taken, as the Semicircle $\frac{1}{4}RP$ to the Semicircle except the Triangle $\frac{1}{4}RP - \frac{1}{4}RD$, that is, as P to $P - D$, ſo C P to C Y: (or, which is the ſame thing, as P to D, ſo C P to P Y:) A Plain through Y Z, parallel to the Baſe, will cut off a Portion of this curved Superficies, adjoining to the Pole, which will be equal to the Triangle A D C. And the ſame being done in the other Quadrants of the curved Surface, the whole ſo cut off, conterminous to the Pole, will be equal to the whole Square inſcribed in the Baſe, and it will be diſtended as was required. *Q. E. F.*

Or thus more conciſely. The curved Surface of the Hemiſphere, (being equal to two great Circles) will be equal to R P. And the Square inſcribed in a great Circle $= 2RR = RD$: And that is to this as P to D. And therefore, becauſe of the Segments of the Surface cut off by the parallel Plains, (being proportional to the Segments of the Axis) taking C P to P Y as P to D, not only the whole Superficies will be equal to R P, but alſo the Portion at the Pole, cut off by the Plain Z Y, will be equal to R D, or to the Square inſcribed in the Baſe. *Q. E. F.*

If it be ſaid that we proceed here upon the Preſumption of the Quadrature of the Circle, or of the Ratio of the Perimeter of the Circle to its Diameter; this is true indeed, but not to be objected here. Becauſe the propoſed Ænigma does not require, that the Portions of the Hemiſpherical Surface which are cut off, or which are call'd the Windows, ſhould be quadruble, but only the remaining Portion. And indeed if both were required, the abſolute Geome-

Geometrical Quadrature of the Circle would be required at the ſame time; which it is plain is not yet found.

As to the manual Conſtruction of the Cupola; upon a plain Baſe, ſituate out of the Baſe of the Hemiſphere, but contiguous to it, whoſe two Sides meet in an Angle at A, within the protracted Right Lines D A, G A, (that there may be a free Proſpect from the Windows on each Side) let a good ſtrong Foundation be laid; ſo that as the Superſtructure riſes higher, its Edge may ſtretch out, ſupported by an Angle, making an Arch of a Circle as D Z, ariſing to the Height Y. And let the ſame be done at the other Angles D, B, G. Laſtly upon theſe Structures, as it were ſo many Columns raiſed to the ſame Height, let the Cupola be placed, ſo excavated within, as the Surface of an Hemiſphere requires. And thus the whole Work is finiſhed as was required. *Fig.* 19.

Otherwiſe. The ſame thing may be done, if any Square Q Q be given, for the Square inſcribed in the Baſe, which is leſs than the curved Hemiſpherical Superficies. For it muſt be made, as R P the curved Hemiſpherical Superficies, to Q Q the given Square, ſo C P the Axis of the Hemiſphere, to P Y the Portion of the Axis that is adjacent to the Pole. Then the Plain Z Y, parallel to the Baſe, will cut off a Portion of the Spherical Superficies which is capable of Quadrature, as being really equal to the given Square Q Q.

The ſame may be perform'd thus, but with more Trouble.

Since a Quadrant of the curved Hemiſpherical Surface D P A, as is ſhewn above, is equal to the Semicircle A B D, and its Segments cut off by Plains parallel to the Baſe are proportional to the Segments of the Axis; in the Quadrantal Arch D P let there be taken P Q an Arch of 60 Degrees; (which was ſuggeſted to me by Mr. *Caſwell.*) Then a Circle Q T S deſcribed with the Pole P will biſect the Axis, becauſe of the Verſed Sine of 60 Degrees being equal to half the Radius: So that it will divide the Quadrant D P A of the curved Hemiſpherical Superficies into two equal Segments. One of theſe, which is the Quadrilineum D Q T S A, will be equal to the Circular Quadrant B C D; and the remaining Trilineum P Q T S will be equal to the Quadrant A C D. Whence if the Bilineum Q R S T be alſo taken away, equal to the Segment of the Circle A D E; the remaining Trilineum P Q R S will be equal to the Triangle A D C. And four ſuch, in the four Quadrants of the Hemiſphere, will be equal to the Square inſcribed in the Baſe. But that Bilineum may be had by what we have formerly proved in the Places above cited. *Fig.* 20.

The ſame may be thus done more univerſally.

Q being any where taken in the Arch D Z, ſo that D Q may not be greater than D Z; and how much the Quadrilineum D Q T S A is deficient from the whole to be taken away, ſo much let the Bilineum Q R S T ſupply. The Remainder will be equal to the Triangle A D C.

And therefore if Q be taken in D, ſo that the Quadrilineum may vaniſh; the Bilineum muſt be taken equal to the whole that is to be taken away.

But if Q is taken in Z, fo that there is no Occafion for the Bilineum, the Quadrilineum will be equal to the whole that is to be taken away.

And all the fame Things, concerning the Quadrilineum and Bilineum, which together compleat the whole to be taken away, are alike to be accommodated, *mutatis mutandis*, if inftead of the Square infcribed in the Bafe, any other Square Q Q is fubftituted, which is not greater than the whole Surface of the Hemifphere.

After this was wrote, and in the Prefs, I was informed, that the learned Mr. *Leibnitz* had given an Anfwer to this Problem, and that it was inferted in the Journals of *Leipfick* for the Month of *June* 1692. Upon which I ftopt the Prefs for fome Weeks, that I might get a Sight of it, which with fome Difficulty I at laft obtained. There I faw that that great Man was of my Opinion, that the Problem is not of that Kind which is called Determinate, but may be folved in a great Variety of Manners, or rather infinite Ways.

The Propofer's Solution demonftrated, by Dr. D. Gregory. N. 207. *p.* 25. Jan. An. 1694.

The Author of this Ænigma has now given us a very ingenious and ready Conftruction of his Problem, in an *Italian* Treatife on the Formation and Menfuration of all Vaults and Cupolas, dedicated to the Grand Duke of *Tufcany*; wherein he has been pleafed to give us his Name, by V V. the laft Difciple of *Galileus:* Whereas before he concealed it by a Tranfpofition of the Letters, as in an Anagram, under the feigned Name of *D. Pio Lifci, pufillo Geometra.*

But now the Ænigma is converted by the Author into the following Problem: *Upon the Surface of an Hemifphere to affign a Portion equal to a given Square.* Which he thus conftructs.

Fig. 22.

Let the Sphere, whofe Axis is equal to the Side of the given Square, be reprefented by the Circle A C B D, which is vertical in the propofed Sphere, whofe Horizontal Diameter is A B, and Center E. Let the Sphere be perforated by two upright Cylinders, whofe common Sections with the Plain A C B D are the Circles B L E G, A H E I, defcribed with the Diameters E B, E A. I fay the Thing is done; that is, from every Hemifphere, for Inftance the upper A C B, four bilinear Figures are taken off by the perforating Cylinders, two on the anterior Side, and two on the pofterior, which are fimilar and fimilarly pofited, fo that the remaining Hemifpherical Superficies is equal to the Square of the Line A B. And becaufe the Hemifpherical Superficies, when the four faid Bilinear Spaces are taken away, reprefents a Sail filled and extended by the Wind, and alfo an Hemifpherical Cupola admitting Light by four Windows, which being conftructed upon a Circular Bafe A E B, refts upon it at the Points A, E, E, B. This he calls, according to a Right he has, *The Quadruple Florentine and Veliform Cupola.*

Then the Author in his Treatife delivers many Things which regard the Practice; how by the Affiftance of the Lathe and Cylindrical Auger to make Models of this, as well as of five other Cupolas. And for this purpofe, he conftructs fome other curious Problems; the Demonftrations of all which are omitted by the Author, but will eafily follow from what is here delivered.

It appears plainly, that the four Windows in the Hemifphere, conftructed as above, are Figures equal, fimilar, and fimilarly pofited. It only remains, that

that we ſhould prove the remaining Hemiſpherical Superficies is capable of a true Geometrical Quadrature.

In the Point E, equal to the Line E A, let a Line be ſuppoſed to be erected, perpendicular to the Plain C A D B; and upon the Periphery ACBD let there be an erect Cylindrical Superficies of the ſame Height. It is commonly known, that a Portion of the Spherical Surface, comprehended between two Plains parallel to the Circle A B C D, is equal to the Portion of the Cylindrical Superficies between the ſame Plains; and that like Portions of theſe Rings, cut off by Plains mutually interſecting one another in the Perpendicular erected at E, are equal alſo. Now by drawing innumerable Plains parallel to the Baſe A C B D, if in the Cylindrical Superficies Parts are conceived to be deſcribed in the aforeſaid Manner, equal to the correſpondent Spherical Parts; that which is repreſented by the Perforation of the Superficies, and taken away oppoſite to it, will be equal to it. So that it appears the remaining Superficies after the Perforation is equal to the remaining Cylindrical Superficies, excepting that which is determined by the ſaid innumerable Plains, and is oppoſite to that which is taken away. Let any Diameter P M be drawn, cutting the Periphery A H E any how in H. Join H A, and through H let R T be drawn perpendicular to A B, and parallel to D C drawn through E, meeting the Periphery A C B D in R and T, and the Periphery A I E in I. Upon the Diameter R T let a Semicircle be drawn, whoſe Periphery is cut by H S and I Q perpendicular to R T, in the Points S and Q. Let the Plain of this Semicircle be conceived to be perpendicular to the Circle A B C D. Whence the Periphery R S Q T will be in the Hemiſpherical Superficies, and the Right Line H S, now perpendicular to the Plain A C B D, will be the Height of the perforating Cylindrical Superficies above the Point of the Baſe H. And the ſame Thing obtains in every Point of the perforating Cylindrical Superficies, *viz.* that its Height to the Superficies of the Sphere above any Point H of the Baſe, is the Right Line H S produced as before. But H S is equal to H A, the Right Sine of the Arch M A, becauſe either of them is a mean Geometrical Proportional between P H and H M; one in the Circle M A P, the other in a great Circle of the Sphere, paſſing through the Points M, S, and P.

If in the Perpendicular to the Plain A C B D erected at E, from E be taken a Right Line equal to H S or H A, and from its Extremity parallel Lines are drawn to P M and U N; the Plain drawn through them will be parallel to the Plain A C B D, and theſe Lines will paſs through the Points S and Q, and being produced as far as the Cylindrical Surface conſcribed about the Hemiſphere, from the Sides of the Cylinder will cut off Right Lines, which will likewiſe be equal to H S or H A, and will include Arches equal and correſpondent to the Arches M N and V P. Now if another Plain Parallel to this at a very ſmall Diſtance be conceived to be drawn in like Manner, by what is already ſhewn, theſe two will mark out in the Cylindrical Surface a Portion of a Ring, equal to the Portion between the ſame Plains, which is taken away from the Hemiſpherical Superficies by its Per-

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foration. Now if the ſame Conſtruction is ſuppoſed to be made at every Point in the Periphery A H E, all the Portions in the Cylindrical Superficies circumſcribed about the Hemiſphere, generated and marked out in the Manner aforeſaid, will be equal to the Spherical Superficies taken away by the Perforation. Therefore the remaining Hemiſpherical Surface will be equal to the remaining Cylindrical Surface, compoſed of all the Right Lines H A, erected at the reſpective Points M, N, V, and P, or to the Figure of the Right Sines of the Semiperipheries A C B, A D B; that is, by what has been long known to Geometricians, to four Times the Square of the Radius A E, or finally to the Square of the Diameter A B. And ſince the two intire Figures contained by the common Section of the aforeſaid perforating Cylindrical Surface with the Surface of the Sphere, are equal to the four Halves of the ſame; it is plain that the remaining Hemiſpherical Superficies A C B, taking away the four Bilinear Spaces as in the Conſtruction aforegoing, is equal to the Square of the Diameter A B. *Q. E. D.*

If the Semiperiphery A H E is ſo inflected, that it may coincide with the equal Quadrant of the Periphery A R C; the Point H will fall upon the Point M, becauſe of equal Arches A H, A M; and H S the Altitude at H of the Cylindrical Surface inſiſting upon A H E, will coincide with its equal H A, the Altitude at M of the Figure of Right Sines erected upon A M C; and the ſame Thing obtains in all other Points. Whence the Curve which is the common Interſection of the Spherical Surface with the Cylindrical Surface, inflected upon the Baſe A H E, though it does not lie in the ſame Plain, yet will coincide with it as ſaid before, and therefore is equal to the Curve that terminates the Figure of Right Sines; that is, to the common Section of the Cylindric Surface erected upon the Quadrantal Arch A R C, with the Plain cutting the Plain of the Baſe in the Right Line B A at half Right Angles; or to a Quarter of the Elliptical Curve whoſe leſſer Axis is A B, and its greater Axis is double in Power to the ſame. Therefore the Perimeter of the Quadrable *Florentine* Sail, conſiſting of four ſuch Arches, is equal to the Perimeter of the aforeſaid Ellipſis.

Moreover it will not be amiſs to add, that the Superficies of two perforating Cylinders within the Sphere, are equal to the Spherical Surface remaining after the Perforation, or to a double *Florentine* Sail, that is, to the double Square of the Diameter. And this appears from hence, that the *Florentine* Sail is equal to four Figures of Right Sines of the Quadrant, and the perforating Superficies is alſo equal to the ſame, becauſe it is congruous with them, if the Inflection is as above.

I ſhall only add one Word more, that the Conſideration of the Figure of Right Sines, (the Parts alſo of which are eaſily changed into Squares) are ſufficient for the Demonſtration of all thoſe Things, which are delivered concerning other Solids wrought by the Turning-Lathe, or perforated by a Cylinder, and their Superficies; by the moſt acute Geometrician V. V. (*Vincentio Viviani*, if I miſtake not) the very worthy Diſciple of *Galilæus*; when he inſtructs us in the Conſtruction and Menſuration of Vaults or Cupolas. Particularly the Surface of the Roman Boat-like Cupola [*Volta a Schifo alla Romana*]

mana] conſiſts of eight Figures of the Right Sines of the Quadrantal Arch, and therefore is equal to the *Florentine* Veliform Cupola. Whence it appears how two Cupolas may be conſtituted upon equal Squares, one of which is ſhut up on all Sides, the other perforated by Windows, each of which is double to the Square of the Baſe.

VII. 1. *Drawing the ſtreight Lines E A, and E B (cutting the Arc A B in G,) and on A G, a Perpendicular E F, (which will therefore paſs to the Center C, beauſe biſecting A G at Right Angles;) the Right-lined Triangle A F E is equal to A D E, the propoſed Portion of the Lunula.* *The Quadrature of the Parts of the* Lunula, *by* Mr. J. Perks, *a little varied, by Dr.* J. Wallis, N. 259. *p.* 411. Dec. An. 1699. *Fig.* 23.

The Demonſtration is to this purpoſe; *viz.* A D B being a Quadrantal Arc; the Angle A G B will be three Halves of a Right Angle; (and its conjunct Angle E G A, half a Right Angle) and that Angle (being external to the Triangle A G E) is equal to the two oppoſite Intervals G E A + E A G. Whereof G E A (becauſe an Angle in the Semicircle A E B) is a Right Angle, and therefore E A G is half a Right Angle, (as are alſo F E G, and F E A) and the three Triangles A F E, G F E and G E A, each of them half a Square. And A G to A E, as $\surd$ 2 to 1, (proportional to the reſpective Radii of the two Circles.) And the like Segments A D G, A E, in their reſpective Circles (as the Squares of their reſpective Radii) as 2 to 1. And therefore the Semiſegment A F D, equal to the Segment A E; and conſequently (one taking from the Triangle as much as the other adds to it) the Portion of the Lunula A D E, equal to the Triangle A F E. *Q. E. D.*

If the Point E chance to be in K (the middle of the Arc A E B), there will be no Interſection at G (the Points G, B, being then coincident, but without any Diſturbance to the Demonſtration): If it happen beyond it, toward B, then G will be on the other ſide; and what is here ſaid of E G B, muſt be accommodated to E G A.

The Ground of the whole Proceſs is plainly this: The Angle A C E, being an Angle at the Center of the greater Circle, but at the Circumference of the Leſſer, the Line C D E (as it paſſeth from C A to C B) doth, in the ſame Proportion, divide the Quadrantal Arc A D B, and the Semicircular A E B: Whence all the reſt doth naturally follow.

2. If you compleat the two Circles, whoſe Arcs contain the *Lunula* of *Hippocrates*, the ſame is true as well of the Points in the other Semicircle A C B, as of thoſe in the Semicircle A E B, and for the ſame Reaſons; as appears in the Scheme annexed, wherein I have mark'd the Points in the Semicircle ACB, (correſpondent to thoſe of Mr. *Perks*, in A E B) with the correſpondent ſmall Letters in the *Roman* and *Greek* Alphabets. *Improved by Dr.* Gregory, *ibid. p.* 414. Dec. An. 1699. *Fig.* 24.

If Mr. *Perks* had made his Conſtruction univerſal, by making both E A and E B meet with the greater Circle, (which he might have done by protracting theſe Lines and the greater Circle till they meet) he might have found that the Portions of the Spaces A ε C M, B H C N, (ſuppoſing M C N parallel to A B) are Quadrable as well as thoſe of *Hippocrates*'s *Lunula*, and that E A γ be-

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ing a ſtreight Line, the Portion A E D of *Hippocrates*'s *Lunula*, is to A ε δ (the Correſpondent of A ε C M) in duplicate Proportion of C ε to A ε; for E R ε (at R the Center of the leſſer Circle) is, in this Caſe, a Right Angle.

Moreover, If you take any Point ε, in the Semicircle A C B, and proceed according to Mr. *Perks*'s Conſtruction univerſaliz'd, as aboveſaid, you will find, on the one ſide, the Trilineum A ε δ (contain'd by the Arcs A ε, A δ, and the ſtreight Line ε δ) equal to the Rectilineal Triangle A ε φ; and on the other ſide, the Trilineum contained by the Arc B ε (the Complement of ε A to the Semicircumference) and the Arc B *d* (the Complement of A δ to the fourth Part of the Circumference) and the ſtreight Line ε *d*, (that is the Trilineum B H C *d* diminiſhed by the Segment C ε) to be equal to the Rectilineal Triangle B ε *f*; and that thoſe two Spaces A ε δ, and the Difference of B H C *d* from the Segment C ε (Parts of the *Lunula* A C B *g* γ A) taken together, are equal to the Triangle A C B, as well as to the two Spaces A E D and B E D, Parts of the *Lunula* of *Hippocrates*.

So that upon the whole it appears, that the two Circles (containing the *Lunula* of *Hippocrates*) being compleated, this *Lunula*, A E B G A, and the other, A C B *g* γ A, make up one Syſtem, and are conjugate Figures.

For, drawing a ſtreight Line C D E, or C ε δ, or ε C *d*, at pleaſure, thro' C, the Center of the greater Circle, and cutting thoſe two Circles, the Space contained within two Arcs of theſe two Circles, and Part of the ſaid ſtreight Line (as A E D, or A ε δ, or B H ε *d*) is equal to the rectilineal Triangle A E F or A ε φ, or B ε *f*, reſpectively.

And ſo it happens, that if this Line going out from C, be on the ſame ſide of the Diameter M N with the *Lunula* of *Hippocrates*, the aforeſaid Space (which receives a perfect Quadrature) is Solitary; ſuch as are the Parts of *Hippocrates*'s *Lunula*, and of the two Spaces, A ε C M, B H C N (which therefore are Parts of the *Lunula*, more nearly relating to one another.)

But if that Line going out from C be on the other ſide of M N, then the Space which is equal to the Rectilineal Triangle is the Difference of two Mixtilineal Figures (the one a Trilineum, the other a Segment of the leſſer Circle), as is aboveſaid; neither of which can be ſquared ſeverally.

All theſe Particulars are plain from Mr. *Perks*'s Demonſtration; which, with a little Variation (ſuch as is uſual in the different Caſes of the ſame Theorem), is applicable to them all; tho' perhaps he was not aware of it.

The like was done (without any Demonſtration) by M. *Tſchirnhauſe*, in the *Acta Lipſiæ* 1687, to this purpoſe: If from any Point E, in the Circumference of the leſſer Circle, we let fall on A B a perpendicular Line cutting it in L, and draw the Line C L; the Triangle C A L is equal to the Portion of the *Lunula* A E D (and conſequently the Triangle C B L equal to the Portion B E D); which I ſhall demonſtrate ſo as the Demonſtration may alſo reach the Portions of the conjugate Space A C B *g* γ A.

For the Triangles A C B, A E F, are Like Triangles, each being the Half of a Square; and therefore, by 19 *El.* 6. the Triangle A C B is to the Triangle A E F in the duplicate Proportion of B A to A E, that is, by 8 *El.* 6. as B A is to A L: But, by 1 *El.* 6. the Triangle A C B is to the Triangle

A C L

A C L as B A is to A L: Therefore, by 9 *El.* 5. the Triangles A C L and A E F are equal: But the Triangle A E F is (by Mr. *Perks*) proved equal to the Portion A E D; and therefore the Portion A E D is alſo equal to the Triangle A C L.

3. On the Center B, Mr. *Caſwell* draws by A a third Circle, which forms another *Lunula* than that of *Hippocrates*; and he doth (very dextrouſly) ſquare the Portions of this *Lunula*; and doth hereby let us into a new Syſtem, which may be purſued in like manner as Dr. *Gregory* hath done that of *Hippocrates*. *By Mr.* Caſwell, *ib. p.* 417.

4. M. *Tſchirnhauſe*, letting fall from E (on A B) a Perpendicular E L, determines the Angle A L C equal to the Portion A D E; which being admitted, we may thus divide the *Lunula* in any given Proportion; if we divide A B at L in ſuch given Proportion, C L will, in the ſame Proportion (becauſe of the common Altitude) divide the Triangle A C B (which is equal to the whole *Lunula)*, and L E (erected at Right Angles on A L B) will determine the Point E; from whence if we draw to C the ſtreight Line E C, this will, at D E, divide the *Lunula* in the ſame Proportion. *By Dr.* Wallis, *ibid.* *Fig.* 25.

Mr. *Perks*, on E D C drawing the Perpendicular A F, determines the Semiquadrate A F E equal to the propoſed Portion A D E; which Semiquadrate is a Like Figure, and alike ſituate to A E as is A C B to A B. *Fig.* 26.

And therefore (becauſe Like Figures are in duplicate Proportion of their reſpective Sides) if we ſo inſcribe A E, as that the Square of A E be to the Square of A B in ſuch given Proportion, the *Lunula* will, at D E, be ſo divided as is required.

And this will hold (if duly applied, according as the different Caſes may require) though E be taken (in the Continuation of the Semicircle) beyond A; for, ſtill like Figures will be in duplicate Proportion of their reſpective ſides, and C E = C D = D E; and the ſame is yet improveable much further.

VIII. If upon B C you take any two Points D, E, and draw the Perpendiculars D H, E M, meeting B A in I and L, and cutting a Portion FGMH, of the *Lunula*; the Solid generated by the Converſion of this Portion about the Axis B C, is equal to a Priſm, whoſe Baſe is I L M H, and Height the Circumference of a Circle whoſe Diameter is B C; and the Solid generated by the Semicircle B K A, is equal to a Priſm or Semicylinder, whoſe Baſe is the Semicircle B K A, and Height the Circumference of a Circle whoſe Diameter is B C. *The Dimenſion of Solids generated by the Converſion of* Hippocrates's Lunula, *by M.* Ab. de Moivre, N. 265. *p.* 624. Jul. An. 1700. *Fig.* 27.

Having biſected B A in R, and B C in P, the Surface generated by the Converſion of the Arc H M about the Axis B C, is equal to $\frac{c}{r}\times\overline{BP\times HM+BR\times DE}$ (ſuppoſing the Ratio of the Radius to the Circumference to be as r to c) and the Surface generated by the Semicircumference B K A is equal to a Rectangle whoſe Baſe is the Sum of that Semicircumference and Diameter B A, and Height, the Circumference of a Circle,

whoſe

whoſe Diameter is B C. As for the Surface generated by the Arc G F, 'tis well known that it is equal to a Rectangle, whoſe Baſe is the Circumference of a Circle whoſe Radius is B C, and Height D E; therefore the Surface generated by the Converſion of the Portion M H F G is known.

Fig. 28. If upon B A, you take any two Points I, L, and draw I N, L V, perpendicular to it, cutting the Quadrant in O and T, and the Circumference in N and V; the Solid generated by the Converſion of the Portion O N V T about the Axis B A, is equal to a Priſm whoſe Baſe is I O T L, and Height the Circumference of a Circle whoſe Diameter is B A.

Having biſected B A in R, and drawn C R meeting the Quadrant in G, the Surface generated by the Converſion of the Arc O T about B A, is equal to $\frac{c}{r}\times\overline{CG\times IL - CR\times OT}$.

Fig. 27. Biſect D E in I; thro' the Center draw S Q, parallel to B C, meeting the Circumference B K A in S, B K parallel to A C in V, and the Lines D H, E M, in N and O; the Solid generated by the Converſion of the Portion F G M N about the Axis A C, is $\frac{c}{r}\times\overline{\tfrac{1}{3}MO^3 - \tfrac{1}{3}NH^3 + PC\times NOMH + CY\times DNOE - \tfrac{1}{3}EG^3 + \tfrac{1}{3}DF^3}$; and the Solid generated by the Segment K B S is $\frac{c}{r}\times\overline{\tfrac{2}{3}VK^3 + PC\times BVKS}$; therefore the Solid generated by the Semicircle B K A about A C, is $\frac{c}{r}\times\overline{PC\times VQAK + PC\times BCQV - \tfrac{1}{3}AC^3 + \tfrac{2}{3}VK^3 + PC\times BVKS}$, which by due Reduction, will be found equal to the Solid generated by the Converſion of the ſame Semicircle about the Axis B C.

Fig. 28. The Solid generated by the Portion O N V T, about the Axis C P, is equal to $\frac{c}{r}\times\overline{\tfrac{1}{3}LV^3 - \tfrac{1}{3}IN^3 - \tfrac{1}{3}QT^3 + \tfrac{1}{3}PO^3 + CS\times PQIL}$.

Fig. 27. From the Points M, H, drop the two Perpendiculars M Z, H W, upon C A prolonged, if need be; the Surface generated by the Converſion of the Arc H M, about the Axis C A, is equal to $\frac{c}{r}\times\overline{PC\times HM - RA\times WZ}$, when the Point Z is next to C; or $\frac{c}{r}\times\overline{PC\times HM + RA\times WZ}$, when the Point W is next to it.

Thoſe that will think it worth their while to beſtow ſome little Pains to find the Demonſtration of this, may ſolve the following Problem.

Any

Any two Conic Sections being given, forming a Lunula by their Intersection, and a Right Line being given by Position, about which, as an Axis, this Lunula is imagined to turn, To find the Solid generated by the Conversion of any of its Parts, cut off by Lines perpendicular to that Axis, or parallel to it, or making any given Angle with it; as also the Surfaces made by that Conversion.

IX. Suppose D P V to be Half of an exterior Epicycloid, V B its Axis, V the Vertex, V L B Half the generant Circle, E its Center; D B the Base, C its Center: Bisect the Arc of the Semicircle V B in L, and on the Center C, thro' L, draw a Circle cutting the Epicycloid in P: Then, I say, the Curvilinear Triangle V L P will be $= BEq \text{ in } \frac{CE}{CB}$; that is, the Square of the Semidiameter of the generant Circle, will be to the Curvilinear Triangle V L P, as C B, the Semidiameter of the Base, to C E; which C E in the exterior Epicycloid is the Sum of the Semidiameters of the Base and Generant; but in the interior Epicycloid D *p u*, it is the Difference of the said Semidiameters. *The Quadrature of a Portion of the Epicycloid, by Mr.* Caswell. N. 217. p. 113. Oct. An. 1695. *Fig.* 29.

COROLL. I.] In the interior Epicycloid, if C E is $\frac{1}{2}$ C B, the Epicycloid then degenerating into a Right Line, the Quadrature of the Triangle *l p u*, will be in effect the same with the Quadrature of *Hippocrates Chius*.

COROLL. II.] If the Semidiameter of the Base is supposed infinite, the Epicycloid then being the common Cycloid, the Area of the said Triangle will be equal to the Square of the Radius of the Generant; and so it falls in with that *Theorem* which *Lalovera* found, and calls *Mirabile*.

The general Proposition from whence I deduced the abovesaid Quadrature, is this, *viz.* The Segments of the generant Circle are to the correspondent Segments of the Epicycloid, as C B to 2 C E + C B. For Example; Suppose F *m* G, the Position of Part of the Generant, when the Point F of the exterior Epicycloid was designed, then the Segment F *m* G *n* is to the Segment D F *n* G, as C B to 2 C E + C B; and consequently the whole Epicycloid to the whole Generant in the same Proportion; which is the only Case demonstrated by M. *de la Hire*.

It follows also, that in the Vulgar Cycloid its Segments are triple of the correspondent Sectors of the Generant; which was first shewn by Dr. *Wallis*.

X. *The Area of the Cycloid or Epicycloid, whether it be primitive, or contracted, or dilated, is to the Area of the generating Circle; and also the Areas of the generated Parts in the same Curves, to the Areas of the analogous Segments of the Circle; as the Sum of the double Velocity of the Center and the Velocity of the Circular Motion, to the Velocity of the Circular Motion.* *A general Proposition for measuring all Cycloids and Epicycloids, by Mr.* Edm. Halley, n. 218. p. 125.

Demonstration. Let any Epicycloid Y P S Q V B be described, by the Revolution of the Circle V L B, upon the Circular Base Y M N B. Let the *Fig.* 30.

Center

Center of the generating Circle be c, and drawing c M K, let the Circle inſiſt upon the Baſe in the Point M, and let S be the delineating Point. Now dividing the Motions, by the Circular Motion firſt let the Point S be tranſferred to R, that the Arch S M may be increaſed by the indiviſible Particle R S; then let the Center c go forward to C. By this Motion the Segment R S M being tranſlated to the Situation Q T N, the Point Q will reach the Curve. It is plain that the Triangle R S M is the Moment or Fluxion of the Area of the Circular Segment: But the Trapezium Q S M N is the Fluxion of the Area of the Curvilinear Space generated at the ſame Time. Now ſince S M, R M, Q M, are ſuppoſed to differ from one another only by a Point, conceive the little Area Q S M N to conſiſt of three Sectors R M S, R M Q, M Q N; and therefore the little Area R M S is to the little Area Q S M N, as the Angle R M S to the Sum of the three Angles R M S + R M Q + M Q N. But the Angles R M Q + M Q N are equal to the Angles M C N + M K N, or to the Angle c M C; becauſe of the Lines R M, Q N inclined to one another in an Angle equal to M K N, and becauſe of the Angle M Q N being half M C N, by *Eucl.* 20. III. Therefore the Angle R M S is to the Angles R M S + c M C, that is, (by the ſame) the Arch $\frac{1}{2}$ R S to the two Arches C c + $\frac{1}{2}$ R S, or R S to 2 C c + R S, as the little Area R S M to the little Area Q S M N; or as the Moment of the Circular Segment Q T N, to the Moment Q S Y M N of the Epicycloid generated at the ſame Time. And as theſe Moments are always in the ſame Ratio, wherever the Point Q is taken, it is plain that the Areas themſelves Q T N, Q S Y M N, generated by theſe Moments, have the ſame conſtant Ratio, or that of the Velocity of the Circular Motion R S, to the double Velocity of the Center adding the Circular Motion, or 2 C c + R S; alſo as the Area V B Z to the Area Q V B N, and therefore the Semicircle V L B to the Curvilinear Space V Q Y N B. So that the Propoſition is manifeſt. Now there is no other Difference in the Manner of demonſtrating, if the generating Circle moves upon an Arch of a concave Baſe, except that the Angle c M C in this Caſe is the Difference of the Angles M C N and M K N. But if the Baſe be a Right Line, then M K N vaniſhing, and becauſe of the Parallels R M, Q N, the Proof becomes eaſier. Now in all theſe Curves there are quadrable Portions, analogous to thoſe Portions in the primary Cycloid, which the learned Dr. *Wallis* has found to be capable of perfect Quadrature: Which eaſily follows from the Premiſes.

With Center K through the Point Q draw the Circular Arch Q Z, and draw Z B cutting off a Segment Z L B, equal to the Segment Q T N; then biſect the Semicircle V B in L, and through the Point L, with the ſame Center K deſcribe the Arch P L, cutting the Epicycloid in P, the generating Circle in T, and the Chords Q N, Z B, in y and X. Now let the Arch V Z $= a$, and its Sine $= s$, the generating Radius $= r$, and the Radius of the Baſe $=$ R. And make the Arch C E, or the Motion of the Center, $= m$. It is plain that the Sector C K E has the ſame Ratio to the Space X y N B, as the Square of K E has to the Difference of the Squares of K L and K B; or as $RR + 2Rr + rr$ to $2Rr + 2rr$; that is, as $R + r$ to $2r$, or K E to B V; and therefore the Rectangle $BE \times CE$, or rm, is equal to the Space $X y N B$. But the Space V Z B is equal to the Rectangle $\frac{1}{2} ar + \frac{1}{2} sr$; and therefore according to our Propoſition it will be, as a to $2m$, ſo is $\frac{1}{2} ar + \frac{1}{2} sr$ to $\frac{mar + msr}{a}$, which is equal to the Curvilinear Space Q V Z L B N Q. From this ſubſtract the Space $X y N B = rm$, and there will remain the Space $QVZXy = \frac{mrs}{a}$. And ſince the Spaces Z X L, $Q y T$ are equal to each other, the Space Q V L T Q will alſo be equal to $\frac{mrs}{a}$. Therefore whenever a to m, or the Circular Motion to the progreſſive Motion of the Center, ſhall be in a given Ratio; the perfect Quadrature of the Curvilinear Spaces Q V L T Q will be given alſo. Alſo the whole Space V P L, to the Square of the Radius B E, will be in the ſame Ratio of the Motions m to a, that is, in every primary Epicycloid in the Ratio of the Radii K E to K B; which is the Propoſition of Mr. *Caſwell*. But the leſſer Spaces Q V L T Q will be to one another as the Sines of the Arches V Z; and the Triangular Spaces Q T P, by the ſame way of arguing, will be as the verſed Sines of the Arches Q T or Z L, and are therefore quadrable. In like manner it may be proved, that the Spaces P Λ ϒ, p L u, p λ ϒ, are always to the Square of the Radius B E (in all theſe Figures) in the aforeſaid Ratio of m to a; and their Portions $p q t$ as the verſed Sines of the intercepted Arches $q t$. But the remaining Segments, as $q t$ ϒ Λ, $q t$ ϒ λ, *&c.* will be as the Right Sines of the Complements of the ſame Arches $q t$.

Now the Ratio of the Velocities m to a is compounded of the Ratio of the Radii K E, B E, and the Ratio of the Angles that are equably deſcribed at the ſame Time C K E, V E Z; and therefore that Ratio of the Angles being given, all the foreſaid Epicycloidal Spaces will alſo be ſquared.

XI. 1. *A Curve is required with this Property, that the two Segments (of a Right Line drawn from a given Point through the Curve,) being raiſed to any given Power, and taken together, may every where make the ſame Sum. We leave it to Analyſts to exhibit a general Solution.* *A Problem propoſed by M.* J. Bernouilli. n. 224. *p.* 387. Jan. An. 1697.

2. The Problem (if I rightly underſtand it) may be thus propoſed. A Curve K I L is required with this Condition, that if a Right Line P K L be any how drawn from ſome given Point or Pole P, meeting the ſame Curve in two Points K and L; the Powers of its two Segments P K and P L, drawn from the given Point P to thoſe Points of meeting, if they are raiſed to an equal Height, (that is, either Squares, or Cubes, or Biquadrates, *&c.*) in every Poſition of that Right Line they may make the ſame Sum, P K q + P L q, or P K$^{cub.}$ + P L$^{cub.}$ *&c.* *Solved by* Iſ. N. *ibid. p.* 389. *Fig.* 31.

Solution. Through any given Point A let an infinite Right-Line be drawn A D B given in Poſition, meeting the moveable Right Line P K L in the Point D; and call A D, x; and P K or P L, y; and let Q and R be Quantities any how compoſed of any given Quantities and the Quantity x; and let the Relation between x and y be denoted by this Equation, $yy + Qy + R = 0$.

$R=0$. And if R be a given Quantity, the Rectangle of the Segments PK and PL will be given. If Q be a given Quantity, the Sum of thoſe Segments (conjoined by their proper Signs) will be given. If $QQ-2R$ is given, the Sum of the Squares ($PKq+PLq$) will be given. If Q^3-3QR is a given Quantity, the Sum of the Cubes ($PK^{cub.}+PL^{cub.}$) will alſo be given. If $Q^4-4Q^2R+2R^2$ be a given Quantity, then the Sum of the Biquadrates ($PKqq+PLqq$) will alſo be given. And ſo on *ad infinitum.* Therefore let it be provided, that R, Q, Q^2-2R, Q, Q^2-2R, Q^3-3QR, *&c.* may be given Quantities, and the Problem will be reſolved. *Q. E. F.*

In the ſame manner Curves may be found, which ſhall cut off three or more Segments having the like Properties. Let there be an Equation $y^3+Qy^2+Ry+S=0$; where Q, R, S, denote Quantities compoſed of any given Quantities whatever, and of the Quantity x any how involved; in which Caſe the Curve will cut off three Segments. Now if S be a given Quantity, the Solid contained by thoſe three will be given. If Q be a given Quantity, the Sum of three ſuch will be given. If $QQ-2R$ be a given Quantity, the Sum of the Squares of three ſuch will be given.

The Uſe of Fluxions *in the Solution of Geometrick Problems; by Mr.* Abr. de Moivre. n. 216. *p.* 52. Mar. An. 1695.

XII. Here you have a Method for the Quadratures of Curvilinear Figures; for the Dimenſion of Solids generated by the Rotation of a Plain, and of their Superficies; for the Rectification of Curves; and for the Calculation of their Centers of Gravity. But before I go any farther, I would have you underſtand, that I make uſe of what the great *Newton* has demonſtrated, in Pag. 251, 252, and 253 of his Philoſophical Principles, about the momentary Increments or Decrements of Quantities, which either increaſe or decreaſe by perpetual Flux; and eſpecially that the Moment of any Power $A^{\frac{n}{m}}$, is $\frac{n}{m}aA^{\frac{n}{m}-1}$.

Therefore the Fluxion $\frac{n}{m}aA^{\frac{n}{m}-1}$ being given, on the contrary we may find the flowing Quantity $A^{\frac{n}{m}}$, firſt by removing a from the Fluxion. Secondly by increaſing the Index of the Fluxion by Unity. And thirdly, by dividing the Fluxion by the Index ſo increaſed by Unity.

The Abſciſs of the Curve in what follows ſhall be denoted by x, its Fluxion by $\dot{x}$; the Ordinate by y, and its Fluxion by $\dot{y}$.

This ſuppoſed, to proceed now to Quadratures, firſt let the Value of the Ordinate be obtained, by means of the Equation expreſſing the Nature of the Curve. Secondly, let this Value be multiplied by the Fluxion of the Abſciſs, and the Rectangle hence ariſing will be the Fluxion of the Area. Laſtly, from this Fluxion of the Area let its Fluent be found, and we ſhall have the Area required.

Let

Let the Equation $x^m = y^n$ be propofed, expreffing the Nature of any Paraboloid, in which the Value of the Ordinate will be $y = x^{\frac{m}{n}}$; which if multiplied by $\dot{x}$, the Rectangle $x^{\frac{m}{n}} \dot{x}$ will be the Fluxion of the Area. Therefore the Area required will be $\frac{n}{m+n} x^{\frac{m}{n}+1}$, or putting y for $x^{\frac{m}{n}}$, it will be

$\frac{n}{m+n} xy$.

Again, let a Curve be propofed whofe Equation is $x^4 + a^2 x^2 = y^2$, (which is the firft among Mr. *Craig*'s Examples) then affuming $x\sqrt{xx+aa} = y$, the Fluxion of the Area will be $x\dot{x}\sqrt{xx+aa}$. Now as it is involved in a Radical Sign, let us fuppofe $\sqrt{xx+aa} = z$; whence $xx+aa = z^2$, and therefore $x\dot{x} = z\dot{z}$; and putting $z\dot{z}$ and z for $x\dot{x}$ and $\sqrt{xx+aa}$, the Fluxion freed from Surds will be $z^2\dot{z}$. This if we bring back to its Original $\frac{1}{3}z^3$, and reftore $\sqrt{xx+aa}$ for z, we fhall have $\frac{1}{3} \times \overline{xx+aa}\sqrt{xx+aa}$ for the Area required.

But that it may farther appear with what Eafe thefe Quadratures may be obtained, I will ftill add one Example more. Let the Equation of the Curve be $\frac{x^2}{x+a} = y^2$; therefore $y = \frac{x}{\sqrt{x+a}}$, and then $\frac{x\dot{x}}{\sqrt{x+a}}$ will be the Fluxion of the Area. Let us fuppofe $\sqrt{x+a} = z$, whence $x = zz - a$, and $\dot{x} = 2z\dot{z}$. Therefore $\frac{x\dot{x}}{\sqrt{x+a}} = 2z^2\dot{z} - 2a\dot{z}$, and therefore $\frac{2}{3}z^3 - 2az$, or $\overline{\frac{2}{3}x - \frac{4}{3}a}\sqrt{x+a}$ will be the Area required.

But it often happens that fome Curves, fuch as the Circle, Ellipfis, or Hyperbola, are of fuch a Nature, that it would be in vain to endeavour to free their Fluxions from Surds; and then the Value of the Ordinate muft be reduced to an infinite Series; then every Term of this Series being multiplied by the Fluxion of the Abfcifs, as above, the Fluent of every Term muft be feparately found, and the new Series thus arifing will exhibit the Quadrature of the Curve propofed.

With the fame Eafe this Method may be accommodated to the Dimenfion of Solids form'd by the Rotation of a Plain, by affuming for their Fluxion the Product of the Circular Bafe into the Fluxion of the Abfcifs. Let the Ratio of the Square to the infcribed Circle be called $\frac{n}{1}$, the Equation belonging to the Circle is $yy = dx - xx$, and therefore $4 \times \frac{dx\dot{x} - x^2\dot{x}}{n}$

is

is the Fluxion of a Portion of the Sphere, and consequently $4 \times \frac{\frac{1}{2} d x^2 - \frac{1}{3} x^3}{n}$ is the Portion itself. The Cylinder circumscribed about this is $4 \times \frac{d x x - x^3}{n}$, and therefore the Ratio of the Portion of the Sphere to the circumscribed Cylinder is as $\frac{1}{2} d - \frac{1}{3} x$ to $d - x$.

The Rectification of Curves will be obtained, if the Hypothenuse of the Right-angled Triangle, whose Sides are the Fluxions of the Abscifs and Ordinate, is considered as the Fluxion of the Curve. But Care must be taken in the Expression of that Hypothenuse, that one of the Fluxions only may remain, and only one of the Indeterminate Quantities, which must be that whose Fluxion is retained. This will be plain from the Examples.

Fig. 32. From the given Right Sine C B. to find the Arch A C, making $AB = x$, $CB = y$, $OA = r$; let C E be the Fluxion of the Abscifs, E D the Fluxion of the Ordinate, and C D the Fluxion of the Arch C A. The Property of the Circle is $2rx - xx = yy$, whence $2r\dot{x} - 2x\dot{x} = 2y\dot{y}$, and therefore $\dot{x} = \frac{y\dot{y}}{r-x}$. But $CDq = \dot{y}\dot{y} + \dot{x}\dot{x} = \dot{y}\dot{y} + \frac{y^2\dot{y}^2}{rr - 2rx + xx} = \dot{y}\dot{y} + \frac{yy\dot{y}\dot{y}}{rr-yy} = \frac{rr\dot{y}\dot{y}}{rr-yy}$. Therefore $CD = \frac{r\dot{y}}{\sqrt{rr-yy}}$. But $\frac{r\dot{y}}{\sqrt{rr-yy}}$ is the Product of $\frac{1}{\sqrt{rr-yy}}$ into $r\dot{y}$, or of $\overline{rr-yy}|^{-\frac{1}{2}}$ into $r\dot{y}$; so that if $\overline{rr-yy}|^{-\frac{1}{2}}$ be reduced to an infinite Series, and all its Terms multiplied by $r\dot{y}$; and if we find the Fluent of every Term, we shall have the Length of the Arch A C.

In a like manner the Arch may be found from the versed Sine being given. Let us resume the Equation found above $2r\dot{x} - 2x\dot{x} = 2y\dot{y}$, which becomes $\dot{y} = \frac{r\dot{x} - x\dot{x}}{y}$. But $CDq = \dot{x}\dot{x} + \dot{y}\dot{y} = \dot{x}\dot{x} + \frac{rr\dot{x}\dot{x} - 2rx\dot{x}\dot{x} + xx\dot{x}\dot{x}}{yy} = \dot{x}\dot{x} + \frac{rr\dot{x}\dot{x} - 2rx\dot{x}\dot{x} + xx\dot{x}\dot{x}}{2rx - xx}$, or reducing all to the same Denominator, and expunging those which destroy one another, 'tis $CDq = \frac{rr\dot{x}\dot{x}}{2rx-xx}$, or $CD = \frac{r\dot{x}}{\sqrt{2rx-xx}}$. Therefore the Length of the Arch A C will easily be obtained, by what has been above delivered.

Sometimes the Fluxion of the Curve is found more easily by a Comparison between the similar Triangles C E D and C B O. For we shall have this Proportion, C B . C O :: C E . C D. That is, for the Circle, $\sqrt{2rx-xx} \,.\, r :: \dot{x} \,.\, \frac{r\dot{x}}{\sqrt{2rx-xx}}$.

The

The Curve-line of the Cycloid may be known in the ſame manner. Let ALK be a Semicycloid, whoſe generating Circle is ADL. Any Point B being aſſumed in the Diameter AL, let BI be drawn parallel to the Baſe LK, meeting the Periphery of the Circle in the Point D. Let the Rectangle AEIB be compleated, and draw FH parallel to EI, and infinitely near it, cutting BI produced in G, and the Curve AK in H. Make $AL = d$, $AB = EI = x$, $GH = \dot{x}$. It is known that the Right Line BG is every where the Aggregate of the Arch AD, and of the Right Sine BD; hence it is manifeſt that the Fluxion IG is the Aggregate of the Fluxions of the Arch AD, and of the Right Sine BD. Now the Fluxion of the Arch AD is found to be $= \frac{\frac{1}{2}d\dot{x}}{\sqrt{dx - xx}}$, and the Fluxion of the Right Sine BD is $= \frac{d\dot{x} - 2x\dot{x}}{2\sqrt{dx - xx}}$. And therefore $IG = \frac{d\dot{x} - x\dot{x}}{\sqrt{dx - xx}}$, and $IHq = IGq + GHq = \frac{dd\dot{x}\dot{x} - dx\dot{x}\dot{x}}{dx - xx}$. Therefore $IH = \frac{\dot{x}\sqrt{dd - dx}}{\sqrt{dx - xx}} = \frac{\dot{x}\sqrt{d}}{\sqrt{x}} = d^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}$; and therefore $AI = 2 d^{\frac{1}{2}} x^{\frac{1}{2}} = 2\sqrt{dx} = 2\,AD$. *Fig.* 33.

This Concluſion may be deduced with very little Trouble from the known Property of the Tangent. For ſince its Particle IH is always parallel to the Chord AD, it cauſes the Trangles IGH and ABD to be ſimilar. Whence AB . AD :: GH . IH. That is, $x . \sqrt{dx} :: \dot{x} . IH = \frac{\dot{x}\sqrt{dx}}{x} = d^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}$.

Now by the Aſſiſtance of the Fluxion IH we may find the Area of the Cycloid. The Fluxion of the Area AEI is the Rectangle $EIG = \frac{dx\dot{x} - x^2\dot{x}}{\sqrt{dx - xx}} = \dot{x}\sqrt{dx - xx}$. But the Fluxion of the Portion ABD does not differ from this; therefore the Area AEI and the correſpondent Portion of the Circle ABD are always equal.

Let AB be the Curve of a Parabola, whoſe Axis is AF, Parameter a. Make $AE = x$, $EB = y$, $AB = z$, $BD = \dot{x}$, $DC = \dot{y}$, $BC = \dot{z}$; and aſſuming an Equation expreſſing the Nature of the Parabola, ſuppoſe $ax = yy$, it will be $a\dot{x} = 2y\dot{y}$, whence $\dot{x} = \frac{2y\dot{y}}{a}$. But $BCq = BDq + CDq$, that is, $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y} = \frac{4y^2\dot{y}\dot{y}}{aa} + \dot{y}\dot{y} = \frac{4y^2\dot{y}\dot{y} + aa\dot{y}\dot{y}}{aa}$, and therefore $\dot{z} = \dot{y}\frac{\sqrt{4yy + aa}}{a}$, or which is all one $\dot{z} = \dot{y}\frac{\sqrt{y^2 + \frac{1}{4}a^2}}{\frac{1}{2}a}$. If therefore *Fig.* 34.

this

this Quantity is reduced to an infinite Series, the Curve A B may thence be known.

Fig. 35. Now it eaſily appears, that if the Hyperbolical Space were given, this would be given alſo, and *vice verſâ*. For $\frac{1}{2} a \dot{z} = \dot{y}\sqrt{yy + \frac{1}{4} aa}$, and therefore $\frac{1}{2} a z$ will be equal to the Space whoſe Fluxion is $\dot{y}\sqrt{yy + \frac{1}{4} aa}$. But this Space is nothing elſe but the exterior equilateral Hyperbola, whoſe Semiaxis A B $= \frac{1}{2} a$, the Abſciſſa A E $= y$, and the Ordinate E G $= x$.

For the Menſuration of the Superficies produced by the Converſion of a Curve about its Axis, there muſt be aſſumed for its Fluxion a Cylindrical Superficies, whoſe Altitude is the Fluxion of the Curve itſelf, and whoſe Diſtance from the Axis is the Ordinate belonging to this Fluxion.

Fig. 32. For Example, let A C be the Arch of a Circle, which by revolving about the Axis A B may generate a Spherical Superficies, which we undertake to meaſure. The Fluxion of the Arch D C is already found to be $\frac{r\dot{x}}{\sqrt{2rx - xx}}$. If we multiply this by the Circumference belonging to the Radius, B C, that is by $\frac{c}{r}\sqrt{2rx - xx}$, (ſuppoſing the Ratio of the Circumference to the Radius to be $= \frac{c}{r}$) we ſhall have the Fluxion of the Spherical Surface $= c\dot{x}$, and therefore the Surface itſelf is $c x$.

As to what belongs to Centers of Gravity, having found the Fluxion of the Superficies or Solid, and multiplying this into its Diſtance from the Vertex, we muſt then return back to the Fluent. This divided by the Superficies or Solid itſelf, will give the Diſtance of the Center of Gravity from the Vertex.

Let the Center of Gravity of all the Paraboloids be to be found. Their Fluxion is thus expreſſed in a general Manner $x^{\frac{m}{n}}\dot{x}$, which multiplied by x becomes $x^{\frac{m}{n}+1}\dot{x}$, whoſe flowing Quantity $\frac{n}{m+2n} x^{\frac{m}{n}+2}$ being divided by the Paraboloids Area $\frac{n}{m+n} x^{\frac{m}{n}+1}$ will give $\frac{m+n}{m+2n} x$ for the Diſtance of the Center of Gravity from the Vertex.

The Center of Gravity in a Portion of a Sphere is found much in the ſame Manner. For its Fluxion $4\,\frac{dx\dot{x} - xx\dot{x}}{n}$ being drawn into x becomes $4\,\frac{dx^2\dot{x} - x^3\dot{x}}{n}$, of which the Fluent is $4\,\frac{\frac{1}{3}dx^3 - \frac{1}{4}x^4}{n}$, which divided by the

Solidity

Solidity of the Portion, that is, by $4\,\frac{\frac{1}{2}dx^2-\frac{1}{3}x^3}{n}$, produces $\frac{\frac{1}{3}d-\frac{1}{4}x}{\frac{1}{2}d-\frac{1}{3}x}x$, or $\frac{4d-3x}{6d-4x}x$, for the Diſtance of the Center of Gravity from the Vertex.

XIII. 1. Prop. I. Prob.] *To find the Relation between the Fluxion of the Axis, and the Fluxion of the Ordinate, in the Curve called Catenaria.*

The Catena; by Dr. Dav. Gregory. N. 231. *p.* 637. Aug. An. 1697.

Fig. 36.

Let F A D be a Chain hanging by its Ends F and D, whoſe loweſt Point, or Vertex of the Curve, is A, its Axis A B perpendicular to the Horizon, and its Ordinate B D parallel to the ſame. We are to find the Relation between B b, or D δ, and $d\,\delta$; ſuppoſing the Point b to be infinitely near B, and $b\,d$ to be parallel to B D, as alſo D δ parallel to B A.

It appears from Mechanicks, that three Powers conſtituted in Æquilibrio have the ſame Ratio as three Right Lines that are parallel to their Directions, or which are inclined to them in a given Angle, being terminated by their mutual Concourſe. So that if D d denotes the abſolute Gravity of the Particle D d, (as would neceſſarily be in a Chain every where equally thick) then $d\,\delta$ will repreſent that Part of the Gravity that acts perpendicularly upon D d, by which it is brought about, (becauſe of the Flexibility of the Chain moving about d) that d D endeavours to reduce itſelf to a vertical Situation. Therefore if $\delta\,d$, or the Fluxion of the Ordinate B D, be ſuppoſed conſtant, the Action of Gravity exerted perpendicularly upon the correſpondent Parts of the Chain d D, will alſo be conſtant, or every where the ſame. Let this be expounded by the Right Line a. Again, by the Mechanicks before cited, D δ, the Fluxion of the Axis A B, will denote the Force, which is exerted according to the Direction of d D, which is equivalent to the aforeſaid Endeavour of the heavy Line d D to reduce itſelf into a Vertical Situation, and which prevents its doing ſo. Now this Force ariſes from the heavy Line D A drawing according to the Direction d D, and therefore *(cæteris paribus)* is proportional to the Line D A. Therefore $\delta\,d$, the Fluxion of the Ordinate, is to δ D, the Fluxion of the Abſciſs, as the conſtant Right Line a is to the Curve D A. *Q. E. F.*

COROL.] If the Right Line T D touches the Catenaria, and meets the Axis B A produced in T, it will be B D : B T : : ($d\,\delta$: δ D : :) a. Curve D A.

Prop. II. Theor.] *If to the Perpendicular A B as an Axis, with Vertex A, an Equilateral Hyperbola A H be deſcribed, whoſe Semiaxis A C is equal to* a; *and to the ſame Axis and Vertex, a Parabola A P be drawn, whoſe Parameter is equal to four Times the Axis of the Hyperbola; and if the Ordinate H B of the Hyperbola be continually produced, till H F is equal to the Curve A P; I ſay the Curve F A D, in which the Points F and D are found, (ſuppoſing B D = B F) will be the Catenaria.*

Make AB$=x$, then B$b=\dot{x}$, and BH$=\sqrt{2ax+xx}$. Whence by the Method of Fluxions, the Fluxion of BH$=\dfrac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}}=mb$. Again, becauſe the Parameter of the Parabola A P is $=8a$, 'tis B P $=\sqrt{8ax}$. Whence np the Fluxion of B P will be $\dfrac{2a\dot{x}}{\sqrt{2ax}}$. So that the Fluxion of the Curve A P $=$

$P p=\sqrt{np\times np+Pn\times Pn}=\sqrt{\dfrac{4a^2\dot{x}^2}{2ax}+\dot{x}^2}=\sqrt{\dfrac{2a\dot{x}^2+x\dot{x}^2}{x}}$, which by multiplying both Numerator and Denominator into $\sqrt{2a+x}$ becomes $\dfrac{2a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}}$. And ſince H E is every where equal to A P, the Fluxion of the Right Line H F, that is $mb+sf$, will be equal to $\dfrac{2a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}}$. But it is already ſound that $mb=\dfrac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}}$. Whence $sf=\dfrac{a\dot{x}}{\sqrt{2ax+xx}}$, which is the Fluxion of B F the Ordinate to the Axis of the Catenaria. Therefore the Fluxion of the Curve A F, or $Ff=\sqrt{sf^2+Fs^2}=\sqrt{\dfrac{a^2\dot{x}^2}{2ax+xx}+\dot{x}^2}=\dfrac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}}$, of which the Fluent is $\sqrt{2ax+xx}$, as juſt now found. Therefore A F $=\sqrt{2ax+xx}$. And it appears that the Fluxion of the Ordinate B F, or $\dfrac{a\dot{x}}{\sqrt{2ax+xx}}$, is to $\dot{x}$ the Fluxion of the Abſciſs A B, as the given Line a to the Curve A F; which is the Property of the Catenaria found above. Therefore the Points of the Catenaria are rightly determined by the foregoing Conſtruction. *Q. E. D.*

COROL. 1.] From the Conſtruction it appears, that B F, the Ordinate of the Catenaria, is equal to the Parabolical Curve A P, taking away B H the correſpondent Ordinate of the conterminate Hyperbola A H.

2. From the Demonſtration it appears, that the Curve of the Catenaria A F is equal to B H the correſpondent Ordinate of the conterminate equilateral Hyperbola. For ſince the Fluxions of theſe Lines are equal, and the Lines themſelves are naſcent at the ſame Time, it is plain they muſt be always equal. Whence the Chain being given, A C or a will be given alſo, as being equal to the Semiaxis of the Equilateral Hyperbola whoſe Vertex is A, and Ordinate equal to the Abſciſs A B of the Chain A D.

3. All Catenaria are ſimilar to one another, ſince they are generated by a like Conſtruction of like Figures ſimilarly poſited. Whence two Right Lines alike inclined to the Horizon, drawn through the Vertices of the Chains, will

will cut off ſimilar Figures, and Portions of the Chains which are proportional to the Right Lines ſo cutting them off.

4. If the Chain Q A D is ſuſpended at the Points Q and D, which are at unequal Heights, the Part of the Curve F A D continues the ſame as if it had been ſuſpended at the Points F and D, which are equally high; becauſe it is all one whether the Point F be fixt to the Horizontal Plain or not.

5. If the Force of the Chain, drawing according to the Direction d D, be denoted by D d; let it be divided (as is commonly known) into the Force $d\,\delta$, according to a Horizontal Direction, and a Force δ D, according to a Vertical Direction. Therefore in the Extremity of the Chain, the Force of approaching directly to the Axis, is to the Force of perpendicular Deſcent in the ſame; or the Part of the ſuſtaining Force acting according to the Direction B D, is to a Part of the ſame acting according to the Direction D δ, as the Semiaxis of the conterminate Hyperbola A H, is to D A the Length of the Chain to the Vertex of the Curve. Whence when the Chain is given, this Ratio is given. And in the ſame Chain ſuſpended more or leſs looſely, that Horizontal Force is as the Axis of the conterminate Hyperbola, ſince D A remains the ſame, when the Extremes of the Chain are equally high.

6. In a Vertical Plain, but in an inverted Situation, the Chain will preſerve its Figure without falling, and therefore will conſtitute a very thin Arch or Fornix: That is, infinitely ſmall, rigid, and poliſh'd Spheres, diſpoſed in an inverted Curve of a Catenaria, will form an Arch, no Part of which will be thruſt outwards or inwards by other Parts, but the loweſt Parts remaining firm, it will ſupport itſelf by means of its Figure. For ſince the Situation of the Points of the Catenaria is the ſame, and the Inclination of the Parts to the Horizon, whether in the Situation F A D, or in an inverted Situation, ſo that the Curve may be in a Plain which is perpendicular to the Horizon; it is plain that it muſt keep its Figure unchanged as well in one Situation as the other. And on the contrary, none but the Catenaria is the Figure of a true and legitimate Arch or Fornix. And when Arches of other Figures are ſupported, it is becauſe in their Thickneſs ſome Catenaria is included. Neither would it be ſuſtained, if it were very thin, and compoſed of ſlippery Parts. From *Corol.* 5. before, it may be collected, by what Force an Arch or Buttreſs preſſes a Wall outwardly to which it is applied. For this is the ſame with that Part of the Force ſuſtaining the Chain, which draws according to a Horizontal Direction. For the Force which in the Chain draws inwards, in an Arch equal to the Chain drives outwards. All other Circumſtances, concerning the Strength of Walls to which Arches are applied, may be geometrically determined from this Theory, which are the chief Things in the Conſtruction of Edifices.

7. Inſtead of Gravity, if any other Power exerts its Force, acting in like manner upon a flexible Line, the ſame Curve will be produced. For Ex-

 ample,

ample, if the Wind be suppofed equable, and fhould blow according to Right Lines parallel to a given Line; the Line thus inflated by the Wind would be the fame as the Catenaria. For fince all Things obtain in this other Force, as we have fuppofed in Gravity, it is evident the fame Line muft be produced.

Fig. 37. Prop. 3. Theor.] *The Hyperbola aforefaid AH remaining, if through A a Right Line GAL be drawn perpendicular to the Axis AB, and a Curve KR be defcribed of fuch a Nature, that BK may be a third Proportional to the Right Lines BA and AC, and to AC be applied a Rectangle AV equal to the interminate Space ABKRLA; the Concourfe F of the Right Lines HB, VG, will be at a Catenaria.*

For by Conftruction 'tis $BK = \frac{aa}{\sqrt{2ax+xx}}$. Therefore the Fluxion of the Space ABKRLA is $BKkb = BK \times B\dot{b} = \frac{aa\dot{x}}{\sqrt{2ax+xx}}$. And fince $BF = \frac{ABKRLA}{AC}$, and AC is given; its Fluxion will be $B\dot{F} = \frac{a\dot{x}}{\sqrt{2ax+xx}}$. But in the Conftruction of the foregoing Propofition, the Fluxion of the Ordinate $B\dot{F} = \frac{a\dot{x}}{\sqrt{2ax+xx}}$. Wherefore this Conftruction comes to the fame as the Conftruction of the foregoing Propofition, and confequently the Point F is at a Catenaria. *Q. E. D.*

COROL.] As in the foregoing Propofition the Catenaria is defcribed from the given Length of the Parabolical Curve; fo in this its Defcription depends on the Quadrature of the Space, in which $xxyy = a^4 - 2axyy$. For BK or $y = \frac{aa}{\sqrt{2ax+xx}}$.

Fig. 36. Prop. 4. Theor.] *The Space AGF contained by the Catenaria AF, and the Right Lines FG, AG, parallel to AB, BF, is equal to the Rectangle under the Semiaxis AC, and DH the Diftance of the Ordinates in the Hyperbola and Catenaria.*

For $\dot{D}H = \dot{B}H - \dot{B}D =$ (by Prop. 2. of this) $\frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}} - \frac{a\dot{x}}{\sqrt{2ax+xx}} = \frac{x\dot{x}}{\sqrt{2ax+xx}}$. Wherefore the Fluxion of the Rectangle under the given Line AC and HD is $\frac{ax\dot{x}}{\sqrt{2ax+xx}} = x \times \frac{a\dot{x}}{\sqrt{2ax+xx}} = fs \times FG =$ the Fluxion of the Space AGF. And fince thefe Figures are nafcent together, it it follows that the Rectangle under A C and D H is equal to the Space AGF. *Q. E. D.*

COROLL.] Hence it follows that the Space F A D, comprehended by the Chain F A D and the Horizontal Right Line F D, is equal to the Rectangle under F D and B A, subtracting the Rectangle under either Axis of the Hyperbola A H, and D H the Excess of the Right Line B H, or of the Curve A D, above the Ordinate B D.

Prop. 5. Theor.] *If the Rectangle L E, equal to the Hyperbolic Space A L H, be applied to the Right Line A L, the Point E will be the Center of Equilibrium of the Catenarian Curve A F D.* Fig. 36.

Let a heavy Curve F A be conceived to be poised upon the Axis G L. From the Doctrine of the Center of Gravity it follows, that the Moment of the Weight F A is expounded by the Surface of an upright Cylinder erected upon F A, and cut off by a Plain passing through G L, making half a Right Angle with the Plain of the Curve. And the Fluxion of this Surface, or $\dot{F}A \times FG$, is equal to the Fluxion of the Space A L H, or $\dot{B}H \times HL$; because $\dot{F}A$ and $\dot{B}H$, as also F G and H L, are equal. And therefore, since they are nascent at the same Time, the said Superficies of the erect Cylinder is equal to the Hyperbolical Space A L H. Therefore this applied to the heavy Line itself A F, or to the Right Line A L which is equal to it, it produces a Breadth A E equal to the Distance of the Center of Gravity from the Axis of Libration G L. Hence E will be the Center of Equilibrium of the Curve F A D, lying equally on each Side of the Axis A B. *Q. E. D.*

COROL. 1.] The Spaces A B H L, B A H, and A G F, are in Arithmetical Proportion. For the Fluxion of the Space A L H $= \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax+xx}} \times x =$

$$\frac{\overline{ax+xx} \times \dot{x}}{\sqrt{2ax+xx}} = \frac{\overline{2ax+xx-ax} \times \dot{x}}{\sqrt{2ax+xx}} = \dot{x}\sqrt{2ax+xx} - \frac{ax\dot{x}}{\sqrt{2ax+xx}} =$$

Fluxion of the Space B A H, lessened by the Fluxion of the Space A G F, by *Prop.* 4. of this. And as these three Figures are nascent at the same Time, it will be $BAH - AGF = (ALH =) BL - BAH$. So that $2\,BAH = BL + AGF$. Whence it follows that the Spaces B L, B A H, and A G F, are in Arithmetical Proportion.

2. The Center of Gravity of the Catenaria descends lower, than that of any other Line of the same Length, and having the same Extremities. For every heavy Body descends as low as it may. And since a Figure descends just so much as its Center of Gravity descends, a heavy flexible Line will so dispose itself, as that its Center of Gravity will be lower than if it assumes any other Figure. And from this Property of a heavy flexible Line, all its other Properties might be easily deduced.

3. If

3. If upon any Curves having the ſame Length, and the ſame Limits D and F as the Catenaria F A D, upright Cylinders were cut by a Plain paſſing through D F; of the Cylindrical Superficies ſo cut off, the greateſt is that which inſiſts upon the Catenaria. For theſe Superficies, if the Angle made by the Plains is half a Right Angle, applied to the Curves themſelves, which in the preſent Caſe are of the ſame Length, produce Breadths equal to the Diſtances of the Centers of Gravity of the Curves from the Right Line D F. Now as in the Catenaria this Diſtance is the greateſt, becauſe of the greateſt Deſcent of the Center of Gravity, the Cylindric Surface to be applied will alſo be the greateſt. And becauſe there is the ſame Ratio of Cylindrical Surfaces cut off by a Plain, containing any Angle with the Plain of the Baſe, as when the ſaid Angle is half a Right Angle, the Propoſition obtains univerſally.

Fig. 36. LEMMA.] *If upon any Ordinate F B perpendicular to the Axis A B of any Curve A F Q, that is deſcribed by the Evolution of another Curve K V, from the correſpondent Point V in K V a perpendicular V R is let fall, meeting the Ordinate in R; if the Fluxion of the Axis A B remains the ſame, the Fluxion of the Fluxion of the Ordinate B F, the Fluxion of the Curve A F, and the Right Line F R, will be continual Proportionals.*

Let the little Right Line F f be produced, till it meets the next Ordinate W ϕ in o. And becauſe by the Hypotheſis F $s = f$ W, it will be $of =$ F f, and therefore $o\phi$ will be the Fluxion of fs, that is, the Fluxion of the Fluxion of the Ordinate. Moreover the Triangles $o\phi f$ and f F R are equiangular, becauſe $o\phi f$ is equal to the Alternate $\phi f r$, and $f o \phi = ($F$fr =)$ FfR, becauſe their Difference Rfr vaniſhes in Reſpect of either of them, ſince Rr is nothing in Compariſon of fr. Therefore it is $o\phi . \phi f :: f$F . FR. But ϕf and fF are equal, ſince they only differ by the Fluxion of each. Therefore $o\phi . f$F $:: f$F . FR. *Q. E. D.*

Fig. 36. Prop. 6. Probl.] *To find the Curve K V, by the Evolution of which the Catenaria A F Q is deſcribed.*

Make A B $= x$, and B F $= y$, as before. Then by *Prop.* 2. of this, 'tis $\dot{y} = \frac{a\dot{x}}{\sqrt{2ax+xx}}$, or $2ax\dot{y}\dot{y} + xx\dot{y}\dot{y} = aa\dot{x}\dot{x}$. Then by *Newton's* Method now in common uſe, 'tis $2a\dot{x}\dot{y}^2 + 4ax\dot{y}\ddot{y} + 2x\dot{x}\dot{y}^2 + 2x^2\dot{y}\ddot{y} = 2a^2\dot{x}\ddot{x} = 0$; (for $\ddot{x} = 0$, becauſe $\dot{x}$ is a conſtant Quantity.) Therefore $\ddot{y} = \frac{-a\dot{x}\dot{y} - x\dot{x}\dot{y}}{2ax+xx} = \frac{\overline{a+x} \times a\dot{x}^2}{\overline{2ax+xx} \times \sqrt{2ax+xx}}$, by ſubſtituting inſtead of $\dot{y}$ its Value $\frac{a\dot{x}}{\sqrt{2ax+xx}}$; (for the Sign — prefixt to the Quantity $\ddot{y}$ only ſhews, that the Place of the Point R in reſpect of F, is oppoſite to the Place of the Point

Point F in respect of B, since the Curve A F Q is concave towards the Axis A B.) And by the second *Prop.* of this, $Ff = \frac{\overline{a+x} \times \dot{x}}{\sqrt{2ax+xx}}$. Wherefore by the foregoing *Lemma*, $FR = \frac{Ffq}{\ddot{y}} = \frac{\overline{a+x}|^2 \times \dot{x}^2}{2ax+xx} \times \frac{\overline{2ax+xx} \times \sqrt{2ax+xx}}{\overline{a+x} \times a\dot{x}^2}$ $= \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a}$. Again, because of the Right-angled Triangles Fsf and F R V, having equal Angles fFs and V F R, because V F s is the Complement of each to a Right Angle, it is $Fs : sf :: FR : VR$, or $\dot{x} : \frac{a\dot{x}}{\sqrt{2ax+xx}} :: \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a}$. V R, which therefore is equal to $a+x$. Therefore this is the Nature of the Curve K V, that if A B be called x, it will be $FR = \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a}$, and $VR = a+x$. *Q. E. I.*

COROL. 1.] $AC : CB :: BH : FR$. For this is the Property of the Right Line F R found above.

2. The Right Line C B is equal to B I or V R; for each of them is equal to $a+x$.

3. The evolving Right Line V F is a third Proportional to the Lines A C and C B. For because of equiangled Triangles fFs and V F R, it is $sF : Ff :: FR : VF$. Or $\dot{x} : \frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}} :: \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a}$. VF, which therefore is equal to $\frac{\overline{a+x}|^2}{a}$. Whence it is $a : a+x :: a+x$. V F, which also is the Radius of a Circle which is equicurved to the Catenaria in the Point F.

4. When the Point F falls in A, or when the Vertex is described by Evolution, that is, when $x=0$, the Value of the evolving Right Line V F, which in this Case is K A, becomes $\frac{\overline{a+x}|^2}{a} = a$. That is, the Point K, where the Curve V K meets the Axis, is as much above the Vertex A of the Chain, as C is depressed below the same. Whence the Diameter of a Circle, equicurved to the Chain at its Vertex, is equal to the Axis of the conterminate Hyperbola A H. Therefore the Chain A D and the Hyperbola A H have the same Degree of Curvature at the Vertex A: For it is generally known that the aforesaid Circle is equicurved to the Equilateral Hyperbola A H in the Vertex A. Also this appears from the Nature of the Chain itself, by what is demonstrated *Prop.* 2. of this. For the nascent Line $FH = AP =$ the nascent $BP = \sqrt{8ax}$ is double to the nascent Line B H or $\sqrt{2ax+xx}$ $=$ (when

$=$ (when x vaniſhes) $\sqrt{2ax}$. And therefore the ſame Point is both in the naſcent Hyperbola and the naſcent Catenaria, That is, the naſcent Hyperbola A H coincides with the naſcent Catenaria A D, and therefore theſe Lines are equicurved at the Vertex A.

5. The Curve K V is a third Proportional to the Right Line A C, and the Curve A F, or the Right Line A L. For from the Nature of Evolution $KV = VKA - KA = VF - KA = \frac{\overline{a+x}^2}{a} - a = \frac{a^2 + 2ax + x^2}{a} - a = \frac{2ax + xx}{a}$. And therefore $a : \sqrt{2ax + xx} :: \sqrt{2ax + xx} . KV$. But $\sqrt{2ax + xx} = AF$, by *Cor.* 2. *Prop.* 2. Whence A C : A F :: A F. K V.

6. The Right Line K I is double to A B. For ſince B I $=$ B C $=$ C A $+$ A B, it will be A I $=$ C A $+$ 2 A B. But A K $=$ C A, by *Corol.* 4. of this. Whence K I $=$ 2 A B.

7. The Rectangle of A C and B R is equal to twice the Hyperbolical Space B A H. For $FR \times AC = \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a} \times a = \overline{a+x} \times \sqrt{2ax+xx} = x \times \sqrt{2ax+xx} + a \times \sqrt{2ax+xx} = AB \times BH + AC \times BH = AB \times BH + AC \times BD + AC \times DH$. Therefore $FR \times AC - BD \times AC = BR \times AC = AB \times BH + AC \times DH$. But by *Prop.* 4. of this, 'tis $AC \times DH =$ Space A G F. And therefore $BR \times AC = ABHL + AGF = 2BAH$, by *Cor.* 1. *Prop.* 5.

Fig. 38. Prop. 7. Theor.] *If in the Logarithmic Curve L A G, whoſe given Subtangent H S is equal to the Right Line* a, *(determined* Cor. 2. Prop. 2. *of this) a Point A be taken, whoſe Diſtance A C from the Aſymptote H P is equal to the Subtangent H S; and from the Points H and P, any how taken in the Aſymptote, equally diſtant from the Point C, if Ordinates H L, P G are erected to the Logarithmic Curve, to half the Sum of which H D or P F are made equal; the Points D and F will be in the Catenaria correſponding to the Right Line A C.*

Make A B $= x$, and therefore C B or D H, the half Sum of the Ordinates H L, P G, will be $a + x$; let the half Difference of the ſame be called y. Then $HL = a + x + y$, and $PG = a + x - y$. And ſince from the Nature of the Logarithmic Curve C A is a mean Proportional between theſe, it will be $aa + 2ax + xx - yy = aa$, and therefore $y = \sqrt{2ax + xx}$. So that $HL = a + x + \sqrt{2ax + xx}$, and $PG = a + x - \sqrt{2ax + xx}$. Therefore the Fluxion of H L, or lm, is $\frac{a\dot{x} + x\dot{x} + \dot{x}\sqrt{2ax + xx}}{\sqrt{2ax + xx}}$.

 And

And becauſe of ſimilar Triangles lm L and L H S, it is L H : H S :: lm : m L. Whence m L or $d\delta$, the Fluxion of B D, is equal to $\frac{a\dot{x}}{\sqrt{2ax+xx}}$. That is, the Curve A D derived from the Logarithmic Curve in the foregoing Manner, is of ſuch a Nature, that if its Axis is called x and its Fluxion $\dot{x}$, the Fluxion of the Ordinate B D will be $\frac{a\dot{x}}{\sqrt{2ax+xx}}$. But this is the very Property of the Catenaria to which a belongs, as demonſtrated in *Prop.* 1. of this. Therefore the Curve F A D above deſcribed is no other but the Catenaria. *Q. E. D.*

COROL. 1.] As by the Help of the Logarithms the Catenaria may be deſcribed, ſo on the contrary by means of the Catenaria, which is conſtructed by Nature herſelf, the Logarithm of a given Number, or rather of a given Ratio, may be found. As ſuppoſing C A to be Unity, whoſe Logarithm is equal to o, let us find the Logarithm of the Number C Q, or of the Ratio between C A and C Q. To the Right Lines C Q and C A let the third Proportional be C V, and let half the Sum of C Q and C V be C B. The Ordinate to the Catenaria from B, that is B D, is the Logarithm required. The Reaſon is plain from the Propoſition.

2. On the contrary, if from the Logarithm given C H or C P the correſpondent Number H L or P G were required, or the Ratio H L to C A, or P G to C A; from H or P let a Perpendicular be raiſed, meeting the Catenaria in D or F; and let C R be made equal to H D or P F, that is to C B, and let it be terminated at the Horizontal Line A R. Then will A R be the Semidifference of the Lines required L H, G P, as H D or C R is their Semiſum, by what is demonſtrated above about the Nature of the Catenaria. (For in three Quantities that are Geometrically proportional, ſuch as H L, C A, P G, the Square of the half-ſum of the Extreams leſſened by the Square of the Mean, is equal to the Square of the half-Difference of the Extreams.) And therefore C R + A R and C R — A R are the Numbers H L or G P, belonging to the given Logarithm C H or C P.

3. From the Demonſtration it is evident, that as H D the half Sum of the Ordinates H L, P G, of the Logarithmick Curve, applied perpendicularly to C H in H, is the Ordinate of the Catenaria; ſo the half Difference of the ſame H L, P G, applied perpendicularly to C A in B, is the Ordinate of the equilateral Hyperbola deſcribed with Center C and Vertex A; and therefore, by *Cor.* 2. *Prop.* 2. of this, is equal to the Catenaria A D: For $y = \sqrt{2ax+xx}$. And ſince it is ſhewn in the foregoing Corollary, that A R alſo is the half Difference of the Right Lines H L, P G, it is plain that A R is equal to the Portion of the Catenaria A D. Whence by the Way a Method is diſcovered, from the Chain A D being given to find C the Center of the conterminate Hyperbola, or that Point in the Aſymptote of the Logarithmick

rithmick Curve G L. For if A R is taken equal to the Chain A D, and from the middle Point of the Right Line B R a Perpendicular to it is raiſed, this will meet B A the Axis of the Chain in the Point required C, as very plainly appears. For thus C R will be equal to C B.

4. Hence alſo it follows, that if the Angle B D T be made equal to A C R, the Right Line D T will touch the Catenaria in D. For thus in the ſimilar Triangles D B T and C A R, it will be D B : B T :: C A : A R, or the Curve A D which is equal to it. And therefore by *Corol. Prop.* 1. of this, D T touches the Catenaria.

5. It alſo follows, that the Space A C H D is equal to the Rectangle of C A and A R. For becauſe A Y D, by *Prop.* 4. is equal to the Triangle under C A and A D—B D = (by *Corol.* 3. of this *Prop.*) A R—A Y = Y R, the Propoſition is plain. And becauſe C A is given, it is evident that the Space A C H D is as the Curve A D, or its Fluxion H *d* is as the Fluxion of this D *d*.

6. If through the Point K, where C R meets H D, a Line K Z is drawn parallel to P H, meeting the Right Line A C in Z, and C E be taken equal to half the Sum of B C, C Z; the Point E will be the Center of Equilibrium of the Curve F A D.

Upon F A D let it be conceived, that the upright Superficies of a Cylinder is erected, and cut by a Plain through P H, at half a Right Angle with the Plain of the Curve F A D. This Superficies will expound the Moment of the Curve F A D, when librated upon the Axis P H; and its Fluxion is $DH \times Dd + PF \times Ff = 2BC \times \dot{AD} = 2\overline{a+x} \times \frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}} = \frac{2aa\dot{x}+4ax\dot{x}+2xx\dot{x}}{\sqrt{2ax+xx}} = \frac{aa\dot{x}}{\sqrt{2ax+xx}} + \frac{aa\dot{x}+ax\dot{x}}{\sqrt{2ax+xx}} + \frac{3ax\dot{x}+2xx\dot{x}}{\sqrt{2ax+xx}}$, of which the Fluent is $a \times BD + a\sqrt{2ax+xx} + x\sqrt{2ax+xx} = CA \times BD + CB \times AD$. Wherefore $CA \times BD + CB \times AD$ is equal to the aforeſaid Cylindrical Superficies, (for they are naſcent together) which is equal to the Moment of the Curve F A D, when poiſed upon the Axis P H. Whence the Diſtance of the Center of Gravity of the Curve F A D from the Point C is $\frac{CA \times BD + CB \times AD}{2AD}$, or $\frac{1}{2}\frac{CA \times BD}{AD} + \frac{1}{2}CB$. Moreover becauſe of Z K parallel to A R, it is A D : B D :: A R : Z K :: C A : C Z, whence $CZ = \frac{CA \times BD}{AD}$, and therefore C E, which by Conſtruction is equal to $\frac{1}{2}BC + \frac{1}{2}CZ$, will be equal to $\frac{1}{2}\frac{CA \times BD}{AD} + \frac{1}{2}BC$. That is, the Center of Gravity of the Curve F A D, and the Point E deter-

mined

mined by the Conſtruction, will be equally diſtant from C; but being in the ſame Right Line, and ſituate the ſame Way, they muſt neceſſarily coincide.

The Coincidence of the Point E, as above determined, with the Center of Equilibrium determined *Prop.* 5. of this, may be thus ſhewn ſynthetically. By *Corol.* 1. *Prop.* 5. 'tis $2\,BAX = AYD + BA \times AR$. Whence $AH + 2\,BAX = ACHD + BA \times AR =$ (by the foregoing *Corol.*) $AR \times CA + BA \times AR$. That is, $BD \times AC + 2\,BAX = AR \times CB$, or $BD \times AC = AR \times CB - 2\,BAX$. Whence $BD \times AC + AD \times BC = AD \times BC + AR \times CB - 2\,BAX = 2\,AD \times BC - 2\,BAX = 2\,AD \times AC + 2\,AD \times AB - 2\,BAX$. And by applying to $2\,AD$, it will be $\frac{1}{2}\frac{BD \times AC}{AD} + \frac{1}{2}BC = AC + \frac{AB \times AD - BAX}{AD} = CA + \frac{ARX}{AR}$. But $\frac{ARX}{AR}$ is the Diſtance of the Center of Equilibrium of the Chain from the Vertex A, by *Prop.* 5. and therefore by the ſame $CA + \frac{ARX}{AR}$ is the Diſtance of the Point E from C, and $\frac{1}{2}\frac{BD \times AC}{AD} + \frac{1}{2}BC$ is the Diſtance of the ſame E from the ſame C, by this *Corol.* Whence it appears that theſe two Determinations of the Point E come to the ſame, becauſe $CA + \frac{ARX}{AR} = \frac{1}{2}\frac{BD \times AC}{AD} + \frac{1}{2}BC$.

7. The Center of Gravity of the Space PFADH is in I, the middle Point of the Right Line CE. For ſince the Center of Gravity of the Fluxion of AD, or Dd and Ff, is diſtant as far again from PH, as the Center of Gravity of the Fluxion of ACHD, or $DHbd$ and $FPpf$, and $\overline{Dd + Ff} \times AC$ is given, equal to $DdbH + FfpP$; it is plain that the Center of Gravity E of the Fluent FAD is as far again diſtant from PH, as the Center I of the Fluent PFADH. But I ſhall prove this otherwiſe, after the manner of the foregoing.

Let an erect Cylinder be ſuppoſed to be raiſed upon the Figure PFADH, and cut off by a Plain paſſing through PH, making half a Right Angle with the Plain of the Baſe; that Solid will repreſent the Moment of the Figure PFADH, when poiſed upon the Axis PH. The Fluxion of this Solid or of the aforeſaid Moment, (that is, the Solids erected upon $PFfp$ and $HDdb$) is produced, if the Moment of the Fluxion, or the Fluxion of the Moment of AD, is drawn into the given Line $\frac{1}{2}AC$. For by *Corol.* 5. of this Propoſition, $HDdb = Dd \times AC$. Wherefore the flowing Moment itſelf is produced by multiplying the Moment of the Curve FAD in reſpect of the Axis PH, determined by the foregoing *Corollary*, that is, $CA \times BD + CB \times AD$ into $\frac{1}{2}AC$; and therefore it will be $\frac{1}{2}AC \times AC \times$

$BD + \frac{1}{2} AC \times CB \times AD$. Now if this be applied to the librated Figure PFADH, or $2 CA \times AD$, by *Cor.* 5. of this *Prop.* there will arise the Distance of the Center of Gravity of the Figure PFADH from the Axis PH, equal to $\frac{1}{2} \frac{CA \times BD}{AD} + \frac{1}{2} CB$, which is equal to half the Right Line CE, as above determined.

8. If through the Point N, where DT the Tangent to the Catenaria in D meets the Line AR, a Right Line be drawn parallel to BC, meeting a Right Line through E parallel to AR in the Point O; this Point O will be the Center of Gravity of the Curve AD. For by *Corol.* 6. the Center of Gravity of the Curve AD is in the Right Line EO. But it will be also demonstrated, that it is in the Right Line NO, and therefore will be in the Point O. Let DA be conceived to be librated about the Axis HL; the Moment of this is the Curve DA, drawn into the Distance of the Center of Gravity from HL, and therefore its Fluxion is $DA \times Hb$, (Hb is the Fluxion of the Distance of the Axis of Libration from the Center of Gravity) which is equal to $\sqrt{2ax + xx} \times \frac{a\dot{x}}{\sqrt{2ax + xx}} = a\dot{x}$. And therefore the Moment of the weighty Curve DA, librated about the Axis HL, is ax. Therefore the Distance of the Center of Gravity from the same Axis is ax applied to AD, or $\frac{AC \times DY}{AR}$. But because DT touches the Catenaria, by *Cor.* 4. of this *Prop.* the Angle BDT or DNY will be equal to ACR. And the Angles at A and Y are right; therefore in the equiangular Triangles RAC and DYN, 'tis $RA : AC :: DY : YN$. Whence $YN = \frac{AC \times DY}{RA}$, that is, YN is the Distance of the Center of Gravity of the Chain AD from the Axis HL; or the said Center is in the Right Line NO.

9. If upon I a Right Line be drawn parallel to AR, meeting ON produced in W; the Point W will be the Center of Gravity of the Space ACHD. For by *Corol.* 7. this Center is in the Right Line IW; and it will be shewn presently, that it is in NW, and therefore is the very Point W. For in the same manner as in the foregoing, the Fluxion of the Moment of the Space ACHD, librated about HL, is shewn to be $ACHD \times Hb = AC \times AD \times Hb = ax\sqrt{2ax + xx} \times \frac{a\dot{x}}{\sqrt{2ax + xx}} = aa\dot{x}$. And therefore the Moment of the Space ACHD, librated about HL, is equal to the Fluent of the Fluxion $aa\dot{x}$, that is to aax. This therefore applied to the Space itself ACHD, or $a\sqrt{2ax + xx}$, gives the Distance of the Center of Gravity of the Space ACHD from HL, that is $\frac{ax}{\sqrt{2ax + xx}} =$

AC×DY

$\frac{AC \times DY}{RA}$. But in the foregoing Corollary it is ſhewn, that $YN = \frac{AC \times DY}{RA}$. Therefore the Center of Gravity of the Space A C H D is in $\overline{N W}$. And by theſe two laſt *Corollaries* the Center of Gravity is found, of any Portion of a Catena that does not reach to the Vertex A, or of any Space of a Catenaria, comprehended by any Portion together with Right Lines.

10. Hence are meaſured the Superficies and Solids produced by the Rotation of a Catenaria, or of any Space comprehended by that and Right Lines, revolving about a given Axis. For the Figure generated by Rotation, as is commonly known, is equal to the revolving Figure drawn into the Periphery that is deſcribed by the Center of Gravity in the Rotation; which Periphery is given, ſince its Radius is given, or the Diſtance of the Center of Gravity from the given Axis. Thus if the Catenaria A D revolves about the Axis A B, the Periphery deſcribed by the Center of Gravity O will be $\frac{\pi}{\varrho} AN$, if $\frac{\pi}{\varrho}$ denotes the Ratio of the Periphery of a Circle to its Semidiameter; and therefore the Superficies produced by the Rotation of the Catena A D, will be $\frac{\pi}{\varrho}, \times AN \times AD = \frac{\pi}{\varrho} \times AN \times AR$. That is, a Circle whoſe Radius is equal in Power to the double of the Rectangle R A N, will be equal to the Superficies produced by the Rotation of the Chain A D about the Axis A B. In like manner it may be ſhewn, that a Solid generated by the Rotation of the Space A C H D about A C, is equal to a Cylinder whoſe Baſe is the aforeſaid Circle, and its Height equal to A C. And ſo may the Superficies and Solids be meaſured, that are produced by the Rotation of theſe Figures about any other given Axis. For when the Centers of Gravity are known, the reſt will eaſily follow.

2. What has been objected by an Anonymous Author, in his Animadverſions upon our Demonſtrations concerning the Catenaria, is this. *Act. Lipſ. M. Feb. An.* 1699. That I have undertaken to demonſtrate, after my Manner, a Matter found out and publiſh'd by others ſeven Years ago. This is true, and I cannot find any thing in this that is Blame-worthy. Thoſe great Men *Huygens*, *Leibnitz*, and *Bernouilli*, have diſcovered and communicated many Properties of the Catenaria, but without Demonſtration. I have contrived Demonſtrations, which was the Thing I undertook to do.

The Animadverſions of . . . . . . by Dr. D. Gregory. N. 259. p. 419. Dec. An. 1699.

But was this Matter (that is, the Nature and primary Properties of the Catenaria) found out and publiſhed by others? Surely that Property of the Catenaria, in *Cor.* 6. *Prop.* 2. was not at all mentioned by others before the Publication of theſe Demonſtrations; although, if I am not miſtaken, it may be reckoned among its primary Properties, is the moſt uſeful of all, and moſt eaſily reduced to the common Purpoſes of Life. From all Ages Architects

have

have made uſe of Arches in publick Buildings, as well for Strength as Beauty. Yet what was the true Geometrical Figure of an Arch, was not known before my Demonſtrations came out.

The firſt Thing he finds Fault with is, that I affirm ſome Things are plain from Mechanicks, which he thinks ſhould have been explained and applied more diſtinctly. As I undertook to demonſtrate ſome Theorems to Geometricians, I did not think it neceſſary to purſue every Thing very minutely. But that I may oblige the Animadverter, I will now demonſtrate that *Lemma*, *Prop.* 1. becauſe I cannot expreſs it more fully than I have already done, in the following Words:

" Three Powers conſtituted in Equilibrio have the ſame Ratio as three " Right Lines, which are parallel to the Directions of the Powers, or are " inclined in a given Angle, and terminated by their mutual Concourſe."

Fig. 39. As ſuppoſe three Powers are in Æquilibrio, that either draw, preſs, or any how act according to three Right Lines P A, P B, P C; and let the three Right Lines E F, F D, D E, be inclined to theſe Directions in any given Angle; that is, let the Angles E A P, F B P, D C P, be equal: I ſay the Powers A, B, C, are to one another as the Right Lines F E, F D, D E.

Let the Right Lines A P, B P, C P, be produced to G, H, K.

In the Quadrilaterum F A B P, becauſe by Hypotheſis the external Angle E A P is equal to the internal and oppoſite Angle P B F, the two internal oppoſite Angles F A P and F B P will be equal to two Right Angles; and ſince all the four internal Angles are equal to four Right Angles, the other two Angles F and A B P oppoſite in the ſame Quadrilaterum, will alſo be equal to two Right Angles. But A P B and B P G make two Right Ones; therefore the Angle F is equal to B P G. In like manner D and E may be ſhewn equal to B P K and A P K.

Now becauſe the three Powers are in Æquilibrio, they are immoveable, and therefore any one of them in reſpect of the two others that remain in Æquilibrio, may be conſidered as a Fulcrum. If B is the Fulcrum, by a moſt known Theorem in Mechanicks, the Power A is to the Power C, as the Sine of the Angle B P K to the Sine of the Angle B P G, that is, as the Sine of the Angle D to the Sine of the Angle F; that is, as the Right Line F E to the Right Line D E. Again, ſuppoſing C the Fulcrum, the Power A is to the Power B, as the Sine of the Angle C P H to the Sine of the Angle C P G, or the Sine of the Angle B P K to the Sine of the Angle A P K; that is, the Sine of the Angle D to the Sine of the Angle E, or as the Right Line F E to F D. Therefore the three Powers A, B, and C, are as the Right Lines F E, F D, and D E. *Q. E. D.*

Fig. 40. We muſt now ſay ſomething about the Application of this Mechanical *Lemma*. If the abſolute Gravity of the little Line *d* D, expounded by *d* D, as ſaid above in *Prop.* 1. is conceived to be collected in its Center of Gravity M, and this heavy Line, by Virtue of its Gravity, endeavours to deſcend according to the Direction M F perpendicular to *d* D; the Power drawing according to M D, which is in Æquilibrio with the ſaid heavy Line, by the forego-

foregoing *Lemma* is to its Momentum or Power drawing according to M F, as δ D is to δ *d*. For the Angle δ D *d*, in which D δ is inclined to M D, is equal to the Angle *d* ε F, in which *d* δ is inclined to M F, for each is the Complement of the Angle *d* to a Right Angle. And this will obtain, as the Animadverter acknowledges, if the aforefaid Weight (as in the vulgar Mechanicks) incumbing upon the Plain M F, is drawn by Help of a Pulley at M, by another Weight incumbing upon M D: Then this will be to that as D δ to *d* δ.

Other Things remaining as before, if the Manner of Application of thefe Powers is changed, fo that to the middle Point M of the flexible Line *d* D, whofe Extremity *d* is fixt, a Weight be applied exerting its Force according to M F; (for in defcending it would defcribe an Arch with Center *d* and Radius *d* M) the Force of this Weight to bend the flexible Right Line at M, would be infinite in refpect of the Force of its abfolute Gravity. And the Force drawing according to M D, which is required in order to prevent the aforefaid bending, would alfo be infinite in refpect of that which was before required, to fupport the Weight M in the Plain M F. So that the Powers which in the former Manner of Application were expounded by *d* δ and δ D, muft now be expounded by infinitely greater Lines, which are ftill proportional to the former. For as before, the Weight M draws according to the Direction M F, and the Power fuftaining it according to M D; and that thefe two are in Æquilibrio appears from the Parts of the Chain being at reft. Therefore the Ratio of thefe remains the fame as before. But the Caufe which extends into a Right Line the flexible Line *d* D, (whofe Extremity *d* is immoveable, and to whofe middle Point M a Weight is applied, which is indeed infinitely little, but whofe Force by this Manner of Application is made infinitely greater, and therefore in the Language of the Animadverter becomes affignable) is the Weight of the Chain D A, which is proportional to its Length. This therefore to the conftant and affignable Quantity *a*, (proportional to the conftant but not affignable Quantity *d* δ) is as D δ to δ *d*. And thus I hope it will appear to the Animadverter, that I have proved the Conclufion to be true, without making any erroneous Pofitions.

XIV. Let A C F be a Semicircle, whofe Diameter is A F; A D E is a Curve Geometrically irrational, whofe Ordinate B D cuts the Semicircle in C. Now let the Quantities be thus reprefented. The Diameter A F $= 2a$, the Abfcifs A B $= y$, the Arch A C $= v$, and the Ordinate B D $= z$. Let $z = rvy^n$ be a general Equation, expreffing the Natures of the Geometrically irrational Curves A D E, in which r denotes any given and determinate Quantity, and n the indefinite Exponent of the indeterminate Quantity y. I fay the Area of the Curve will be

The Quadrature of Figures Geometrically irrational, by Mr. J. Craig. N. 232. p. 703. Sept. An. 1697. Fig. 41.

$$ABD = \frac{rvy^{n+1}}{n+1} - qv + \sqrt{ry - yy} \text{ into } \frac{ra}{\overline{n+1}^2} y^n + \frac{2nra^2 + ra^2}{n \times \overline{n+1}^2}$$

$$y^{n-1} + \frac{aA \times \overline{2n-1}}{n-1} y^{n-2} + \frac{aB \times \overline{2n-3}}{n-2} y^{n-3} + \frac{aC \times \overline{2n-5}}{n-3} y^{n-4} + \frac{aD \times \overline{2n-7}}{n-4} y^{n-5} + \frac{aE \times \overline{2n-9}}{n-5} y^{n-6}, \text{ \&c.}$$

Concerning this infinite Series the following Things are to be obſerved. (1.) That the Capitals A, B, C, D, E, *&c.* denote the Coefficients of the Terms immediately preceding. Thus $A = \frac{ra^2 \times \overline{2n+1}}{n \times \overline{n+1})^2}$, $B = \frac{aA \times \overline{2n-1}}{n-1}$, $C = \frac{aB \times \overline{2n-3}}{n-2}$, and ſo on. (2.) That if the Exponent n repreſents any integer affirmative Number, or is $= 0$; or if $2n$ be an odd Number, then the Quadrature of the Space A B D is exhibited by a finite Quantity, becauſe of the Series breaking off in theſe Caſes. (3.) That q repreſents the laſt Term ſo breaking off. (4.) That all thoſe Figures in which the Series breaks off have a Portion Geometrically quadrable, which is eaſily aſſigned by the Series itſelf. For if the Abſciſs y is taken equal to $r^{\frac{1}{n+1}} \overline{\times nq+q})^{\frac{1}{n+1}}$, the Area belonging to this Abſciſs will be Geometrically quadrable. (5.) That only the irrational Term $\sqrt{2ay-yy}$ is to be multiplied into the Terms that follow it.

Example 1.] Let $z = v$. Now becauſe in this Caſe 'tis $r = 1$, $n = 0$, therefore $\frac{ra}{\overline{n+1})^2} y^n$ is the laſt Term breaking off. Therefore $q = a$, and $ABD = vy - av + a\sqrt{2ay-yy}$. Therefore by *Note* 4, if there be taken the Abſciſſa $y = a$, that is, if the Ordinate paſſes through the Center of the Circle, the Portion belonging to it will be Geometrically quadrable: For then Area $= aa$, or the Square of the Radius.

Example 2.] Let $z = \frac{vy}{a}$. Becauſe in this Caſe $r = \frac{1}{a}$, $n = 1$, therefore $\frac{2nraa + raa}{n \times \overline{n+1})^2} y^{n-1}$ is the laſt Term breaking off, ſo that $q = \frac{3a}{4}$, whence $ABD = \frac{vy^2}{2a} - \frac{3av}{4} + \frac{y+3a}{4}, \sqrt{2ay-yy}$, wherefore by *Note* 4, if we take $y = \frac{\sqrt{3aa}}{2}$, the Geometrically quadrable Area belonging to this Abſciſs will be $\sqrt{\sqrt{6a^4} - \frac{3a^2}{2}} \times \frac{\sqrt{3a^2}}{32} \times \frac{3a}{4}$.

Example

Example 3.] Let $z = \frac{v y^2}{a a}$, in this Case $r = \frac{1}{a a}$, $n = 2$, and therefore $\frac{a A \times \overline{2 n - 1}}{n - 1} y^{n-2}$ is the last Term breaking off. Therefore $q = \frac{5 a}{6}$; so that by the infinite Series it will be

$$A B D = \frac{6 v y^3 - 15 a^3 v + \overline{2 a y^2 + 5 a^2 y + 15 a^3} \times \sqrt{2 a y - y y}}{18 a^2}.$$ And therefore by *Note* 4, if we take $y = \sqrt[3]{\frac{5 a^3}{2}}$, the Area belonging to this Absciss will be Geometrically quadrable; that is, the Area $\frac{2 a y^2 + 5 a^2 y + 15 a^3}{18 a}$ $\times \sqrt{2 a y - y y}$.

Secondly, let A C F be a Parabola, whose Axis is A E, Vertex A, and Parameter B A. And let A D G be a Curve Geometrically irrational; whose Ordinate B D cuts the Parabola in C. Make the Abscis A B $= y$, the Ordinate B D $= z$, and the Arch of the Parabola A C $= v$. And let the general Equation, expressing the Natures of an Infinity of irrational Curves, be $z = r v y^n$, in which r denotes a given determinate Quantity, and n the indefinite Exponent of the indeterminate Quantity y. I say the Area *Fig.* 4.

$$A B D = \frac{r y^{n+1} \times v}{n + 1} - q v + \sqrt{2 a y + y y} \text{ into } \frac{r}{\overline{n + 2} \times \overline{n + 1}} y^{n+1}$$
$$- \frac{r a}{\overline{n + 2} \times \overline{n + 1}|^2} y^n + \frac{r a a \times \overline{2 n + 1}}{n \times \overline{n + 2} \times \overline{n + 1}|^2} y^{n-1} - \frac{a A \times \overline{2 n - 1}}{n - 1} y^{n-2}$$
$$+ \frac{a B \times \overline{2 n - 3}}{n - 2} y^{n-3} - \frac{a C \times \overline{2 n - 5}}{n - 3} y^{n-4}, \text{ \&c.}$$

Concerning this Series these Things are to be observed. (1.) That the great Letters A, B, C, *&c.* represent the Coefficients of the preceding Terms respectively. (2.) That if the Exponent n is an affirmative Integer, or equal to nothing, or likewise if $2 n$ be an odd Number, then the Quadrature may be exhibited by a finite Number of Terms, the Series breaking off of itself. (3.) Then is q equal to the last Term breaking off. (4.) That of the Terms multiplying the Quantity $\sqrt{2 a y + y y}$, the last breaking off is to be doubled. (5.) That all those Figures in which n is an Integer positive and odd Number, or more generally, all the Figures in which the final Term has the Sign $+$, have a Portion Geometrically quadrable, which is easily assigned by the Series, by taking the Abscis as in *Not.* 4. of the foregoing Series.

Example 1.] Make $z = v$. Because in this Case $r = 1$, $n = 0$, therefore the Term last breaking off is $- \frac{r a}{\overline{n + 2} \times \overline{n + 1}|^2} y^n$, whence $q = - \frac{a}{2}$ by by *Not.* 3. And because in this Case $-\frac{a}{2}$ is the final Term, therefore $-a$ is the last Term to be multiplied into $\sqrt{2ay+yy}$, by *Not.* 4. and therefore $ABD = vy + \frac{av}{2} + \sqrt{2ay+yy} \times \overline{-\frac{1}{2}y - a}$.

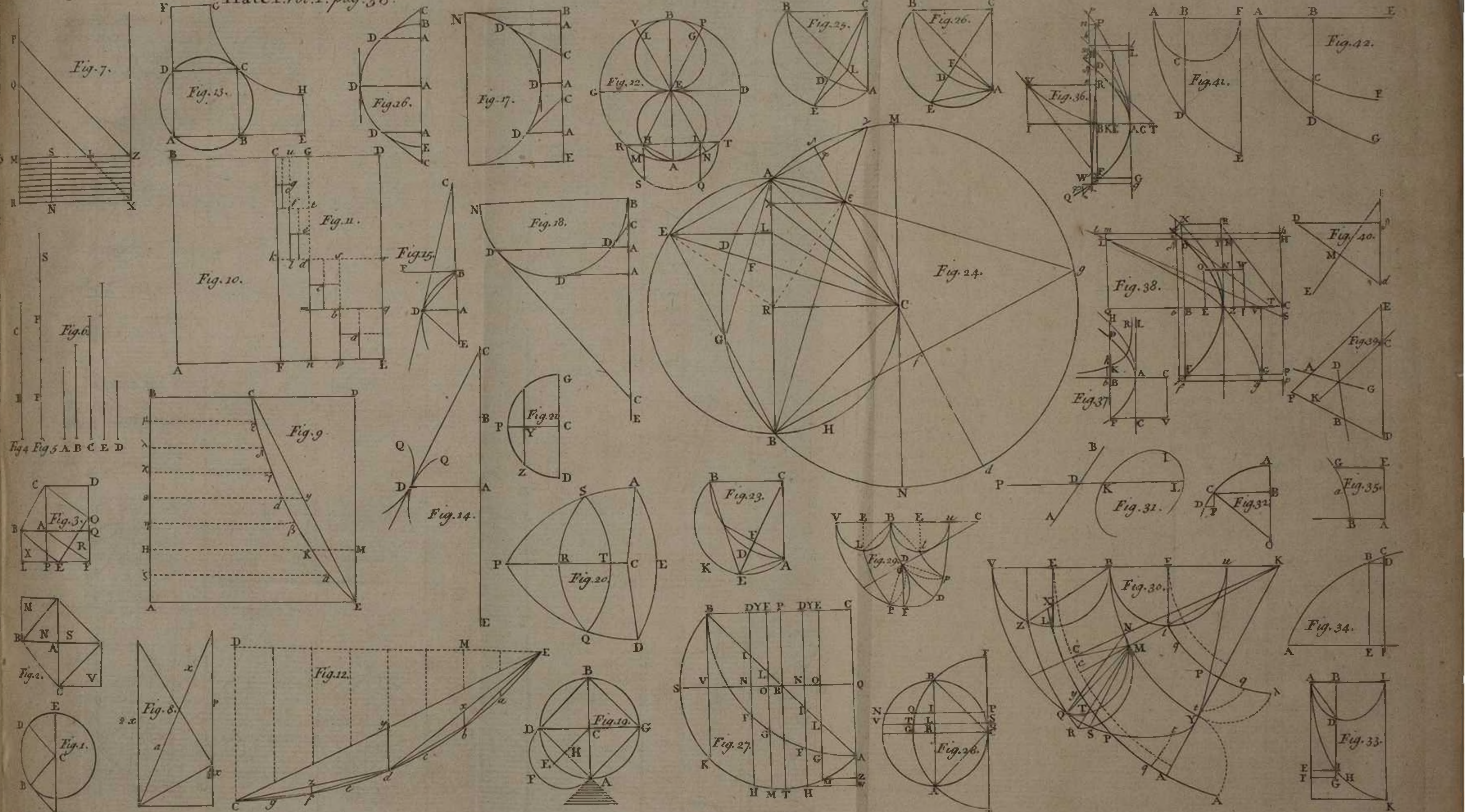
Example 2.] Let $z = \frac{vy}{a}$. Because in this Case $r = \frac{1}{a}$, $n = 1$, therefore the Term breaking off is $\frac{raa + 2\overline{n+1}}{n \times \overline{n+2} \times \overline{n+1}|^{2}} y^{n-1} = \frac{a}{4}$. Whence $q = \frac{a}{4}$, and $\frac{a}{2}$ is the last Term to be multiplied into $\sqrt{2ay+yy}$. Therefore $ABD = \frac{vyy}{2a} - \frac{av}{4} + \sqrt{2ay+yy} \times \overline{-\frac{yy}{6a} - \frac{y}{12} + \frac{a}{2}}$. And if we take $y = \sqrt{\frac{aa}{2}}$, the quadrable Area belonging to this Abscis will be $\frac{1}{12}$ $\sqrt{\sqrt{2a^4} + \frac{aa}{2}} \times \overline{5a - \sqrt{\frac{aa}{2}}}$.

N. 235. p. 785. Fig. 41. *Thirdly.*] Let ACF be a Semicircle, ADE a Curve Geometrically irrational, whose Ordinate BD cuts the Semicircle in C. Let the Quantities be represented as before, that is, the Diameter $AF = 2a$, the Abscis $AB = y$, the Arch $AC = v$, the Ordinate $BD = z$. And let the Equation expressing the Natures of the Curves ADE be $z = rv^2y^n$, in which r denotes any given determinate Quantity, and n the indefinite Exponent of the indeterminate Quantity y. I say the Area is

$$ABD = rv^2y^{n+1} - qv^2 + v\sqrt{2ay-yy} \text{ into } \frac{2ra}{\overline{n+1}|^{2}}y^n + \frac{2ra^2 \times \overline{2n+1}}{n \times \overline{n+1}|^{2}}y^{n-1} + \frac{aA \times \overline{2n-1}}{n-1}y^{n-2} + \frac{aB \times \overline{2n-3}}{n-2}y^{n-3} + \frac{aC \times \overline{2n-5}}{n-3}y^{n-4} + \frac{aD \times \overline{2n-7}}{n-4}y^{n-5}, \text{ \&c.} - \frac{2ra^2}{\overline{n+1}|^{3}}y^{n+1} - \frac{2ra^3 \times \overline{2n+1}}{n^2 \times \overline{n+1}|^{2}}y^{n} - \frac{a^2A \times \overline{2n-1}}{\overline{n-1}|^{2}}y^{n-1} - \frac{a^2B \times \overline{2n-3}}{\overline{n-1}|^{2}}y^{n-2},$$
&c.

Of this Theorem these Things are to be observed; (1.) That it is compounded of two infinite Series, of which the first (connected by the Sign +) is multiplied into $v\sqrt{2ay-yy}$; but the Terms of the latter (affected with the Sign —) are absolute. (2.) That in the former the great Letters A, B, C, D, &c. denote the Coefficients of the Terms that precede them: Also in the latter they have the same Values as in the former. (3.) That the

Plate I. *Vol. 1. pag. 58.*



the Quadrature is exhibited by a finite Quantity, when n is an Integer positive Number, or equal to nothing, or when $2n$ is an odd Number. For in these Cases both the Series break off. (4.) That $2q$ is equal to the last Term breaking off of the former Series.

Example 1.] Let $z = \frac{v^2}{a}$. Because in this Case $n = 0$, $r = \frac{1}{a}$, therefore it will be Area A B D $= \frac{yv^2}{a} - v^2 + 2v\sqrt{2ay - yy} - 2ay$.

Corol. The intire Figure A F E is equal to a double Square, the Side of which is A C F, taking away the Square of the Diameter.

Example 2.] Let $z = \frac{yv^2}{a^2}$. Because in this Case $n = 1$, $r = \frac{1}{a^2}$, therefore the Area will be A B D $= \frac{y^2v^2}{2a^2} - \frac{3}{4}v^2 + v\sqrt{2ay - yy} \times \overline{\frac{v}{2a} + \frac{3}{2}}$ $+ \frac{1}{4}yy - \frac{3ay}{2}$.

Example 3.] Let $z = \frac{y^2v^2}{3a^3}$. Because in this Case $n = 2$, $r = \frac{1}{a^3}$, the Area will be A B D $= \frac{y^3v^2}{a^3} - \frac{5}{6}v^2 + v\sqrt{2ay - yy} \times \overline{\frac{2y^2}{9a^2} + \frac{5y}{9a} + \frac{5}{3}}$ $- \frac{2y^3}{27a} - \frac{5y^2}{18} - \frac{5ay}{3}$.

XV. Let O N F be a Logarithmick Curve, whose Asymptote is A R, in which let such a Point A be taken, that its first Ordinate A O may be equal to the Subtangent, or to Unity. The Curvilinear Space A O N M is required, comprehended by the two Ordinates A O, and M N, by the Absciss A M, and by the Logarithmic Curve O N. *The Quadrature of the Logarithmick Curve, by Mr. J. Craig. N. 245. p. 373. Oct. An. 1698. Fig. 43.*

From O draw O E parallel to A M, cutting M N in E. I say that the Rectangle of the Segments M E, E N, is equal to the Space required.

Demonstration.] Make the Ordinate M N $= z$, the Subtangent A O or M E $= s$, and to the Axis A R let another Curve H G Q be constructed, whose Equation is $2sz = xx$, where its Ordinate G M $= x$. I say it is the Quadratrix of the Logarithmic Curve, according to the Foundation of my Method: That is, its Subnormal is equal to the respective Ordinate of this, as according to the Calculation belonging to that Method may appear. Therefore according to what I have explained elsewhere, if to G be drawn G C perpendicular and equal to the Line G M, and also H D parallel to G C, meeting the Lines G M, C M, in B and D; it will be Trapezium

GBDC = AONM. But GBDC = GMC − BMD = $\frac{1}{2}xx - \frac{1}{2}$ BMq = sz − $\frac{1}{2}$ HAq. But HA = $\sqrt{2}$ AOq by the Nature of the Curve HGQ. Therefore GBDC = sz − AOq = AO × $\overline{MN - AO}$ = ME × $\overline{MN - ME}$ = ME × EN. Therefore also AONM = ME × EN. *Q. E. D.*

A Quadratix to the Circle; being the Curve describ'd by its Equable Evolution, by .... N. 260. p. 445. Jan. An. 1700. *Fig.* 44.

XVI. 1. By the Equable Evolution of a Circle, I mean such a gradual Approach of its Periphery to Rectitude, as that all its Parts do together, and equally, evolve or unbend; or so that the same Line becomes successively a less and less Arc of a reciprocally greater Circle.

2. Let AHKA be the Periphery of a Circle, AE a Tangent to the Point A. Let this circular Line be suppos'd cut or divided at A, and then to unbend (like a Spring,) its upper End remaining fixed to its Tangent AE, whilst the other Parts do equally evolve or extend themselves through all the Degrees of less Curvature (as in ABD, AMC, *&c.*) till they become streight in Coincidence with the Tangent AE.

3. Let AMC be the evolving Curve in any middle Position between its first and last. Join the fix'd End A, and the moving End C, by the Chord-line AC, intersecting the first Circle at H; I say, that AMC is a like Segment to A*n*H, cut off in the first Circle by the Chord AH: For, by the Supposition, AMC is the Arc of a Circle, having AE a Tangent common both to it and A*n*H; and both Arcs are terminated in the same Right-Line AC.

4. Hence the Curve ADCE (described by the moving End of the Periphery in its Evolution) may be thus constructed: Let the Circle AHKA be by Bisections divided into any number of equal Parts; let H be one of the Points of such Division: Then say, As the Number of equal Parts in the Arc A*n*H, is to the Number of Parts in the whole Periphery AHKA; so is the Chord AH, to a fourth Line, which let be AC in AH produc'd. So is C a Point in the Curve ADCE.

5. *Dem.* Upon AC describe AMC, an Arc like to the Arc A*n*H. Whence AH : AC :: A*n*H : AMC. But by Construction, AH : AC :: A*n*H : Periph. AHKA; therefore is the Arc AMC equal to the whole Periphery AHKA, and like to the Arc A*n*H: Consequently AMC represents the evolving Periphery, in a Position like to the Arc A*n*H, and C is the describing Point.

6. After the same manner may be found other Points, thro' which the Curve may be drawn: But here (as in the old *Quadratrix* of *Dinostratus*) the Point E cannot be precisely determined; but the Curve may be brought so near it, that its Flexure or Tendency will so lead to the Point E, that AE shall be near enough to the Truth for common Uses.

7. Supposing the Point E found, a Tangent to any Point of the Curve may be drawn; and supposing a Tangent drawn, the Point E may be determined; the Property of the Tangent being this, that supposing RT a Tan-

gent to the Point C, and C A, C E, drawn from C to each End of the rectify'd Circle, the Angle A C T (the leſſer Angle that A C makes with the Tangent) is equal to A C E, the Angle made by the two Lines drawn from C.

8. Let *c* be a Point in the Quadratrix indefinitely near to C; and draw A *c* interſecting A H K A in *h*, and A M C in *o*. To A *c* as a Chord, draw the Arc A *m c*, like unto the Arc A *n h*: To the Point C of the Arc A M C draw the Tangent C L = A E, and join L A; ſo is *o* C an indefinitely little Particle of the Arc coincident with its Tangent.

9. Becauſe of the like Segments A *n h* A, A M *o* A, A *m c* A, as Chord A *c* to Chord A *o*, ſo is Arc A *m c* (= A M C) to Arc A *m o*: Or, A *c* : A *o* :: A *m c* (= A M C) : A M *o*; and dividing, A *c* — A *o* (= *c o*) : A *o* :: A *m c* — A M *o* (= C *o*) : A M *o*: That is, *c o* : A *o* :: C *o* : A M *o*; and alternately, *c o* : C *o* :: A *o* : A M *o*. Put A C for A *o*, and A M C for A M *o* (as differing infinitely little) and then 'tis *c o* : C *o* : : A C : A M C. But by *Conſtruction* C L = A E = A M C, whence *c o* : C *o* :: A C : C L; and the Angle L C A = C *o c*, (*o c* being infinitely near to A C, is therefore parallel to it;) and therefore C *o c*, A C L, are like Triangles.

10. Becauſe of C L = A E, Angle E A C = L C A, (C L and E A being Tangents to the two ends of the ſame Circular Arc A M C, make equal Angles with its Chord A C) and A C common to both, the Triangles E A C, and A C L, are like and equal: therefore are all three C *o c*, A C L, E A C, like Triangles. Whence it follows, That the Angle A C E (in the Triangle E A C) is equal to the Angle *o c* C (in the Triangle *c o* C) but *o c* C = A C T, becauſe *o c* and A C are parallel; therefore the Angle A C E = A C T. *Q. E. D.*

XVII. In a certain Epiſtle of mine, in Vol. 3. of my Mathematical Works, among other Methods for Quadratures are to be found theſe two. One I call the Method of Convolution and Evolution, the other the Method of Complication and Explication. By Help of theſe I ſhew, which is the ſimpleſt Manner of meaſuring all Curve Figures, and particularly the Cycloid.

The Dimenſions of a Sphere and Cylinder compar'd; by D. Wallis. N. 263. *p.* 547. An. 1700.

By a like Artifice may be ſhewn how to compare the Sphere and Cylinder, which *Archimedes* thought fit to chuſe for his Monument.

If to the Baſis P, equal to the Circumference of a Circle, a Height R be aſſumed equal to Radius, there will be made a Rectangular Parallelogram = R P. This may be conceived as compoſed of an infinite Number of ſmall Parallelograms, of the ſame Height, according to the received Method of Indiviſibles. *Fig.* 45.

Now if the Vertices of all theſe are conceived to be contracted into one Point, ſo that of thoſe minute Parallelograms as many Triangles may be made, having the ſame Baſis and an equal Height; each will be half of each of the others, and therefore all of all; and the Baſe being bent into the Circumference of a Circle, a Circle will be made whoſe Radius is R and Center C, which therefore is half the Parallelogram, or $\frac{1}{2}$ R P. *Fig.* 46.

This

This is *Archimedes*'s Dimension of a Circle, which is equal therefore to a Right-angled Triangle, one of whose Sides about the Right Angle is equal to the Periphery, and the other to the Radius of the proposed Circle. For $\frac{1}{2}$ R, or half the Altitude of the Triangle, drawn into P the Base, exhibits the Magnitude of that Triangle $\frac{1}{2}$ R P, which is equal to the Circle. And the same may be accommodated to the Circular Sector, taking the Arch A instead of the Periphery P.

Fig. 47. Again, if to that Parallelogram = R P, as a Base, be taken in like manner an Altitude R, in order to a Hemisphere; there will be made a Parallelepiped = R R P. This in the same manner may be conceived as composed of an infinite Number of small Parallelepipeds of the same Height, insisting upon the minute Areas of that Plain; of all which the common Altitude is R, and the Aggregate of Bases = R P. Now if this Parallelogram, the Magnitude R P continuing, be supposed to be bent into a Cylindrical Surface (whose Base P is now bent into the Periphery of a Circle, and whose Altitude is R) that those minute Parallelepipeds may be changed into so many Wedges, or Prisms with Triangular Bases, each of which are half their respective Parallelepipeds, and therefore all are half of all; having for their Vertices so many Points C, or minute Lines, in the Axis of the Cylinder, and thus filling it up; the Cylinder will become half the Parallelepiped, or $\frac{1}{2}$ R R P.

Or in order to come at the intire Sphere, if on each Side is taken the Altitude R, so that the whole Altitude may be D = 2 R, and if a Convolution be made in like manner, a Cylinder will be produced as before, consisting of Wedges or Prisms infinite in Number, having their Points or Vertices in the Axis of the Cylinder, which will be equal to R R P = $\frac{1}{2}$ R P x 2 R, equal to the Product of $\frac{1}{2}$ R P, or the Circular Base, into the Altitude 2 R: Or which is the same Thing, it will be equal $\frac{1}{2}$ R x 2 R P, or equal to the Product of $\frac{1}{2}$ R, the half of the common Altitude of the Wedges, into the Aggregate of the Bases 2 R P.

Which Aggregate of the Bases is the curved Cylindric Superficies itself, which is equal to P x 2 R, or to the Product of P the Periphery of the Circular Base drawn into the Altitude 2 R, or equal to $\frac{1}{2}$ R P x 4, four of the great Circles of the Sphere. To which if we add the two opposite Circular Bases, there will be made the whole Superficies of the Cylinder circumscribed to the Sphere, equal to six great Circles, $\frac{1}{2}$ R P x 6 = 3 R P. And the Magnitude of the Cylinder, = R R P = $\frac{1}{2}$ P P x 2 R, equal to the Product of the Circular Base $\frac{1}{2}$ R P drawn into the Altitude 2 R, as before.

Now if the Vertices of all these Wedges that constitute the Axis of the Cylinder, are conceived to be contracted into one Point, so that these Wedges or Prisms may now become so many Pyramids, being on the same Bases and the same Height; each will be of each, and therefore all of all, in a Proportion subsesqui-tertial or as $\frac{1}{3}$ to $\frac{1}{2}$; and the Superficies which before was curved Cylindrical, will now become Spherical because of all its Points being equally remote from the Center, the Aggregate of the Bases remaining as before = 2 R P, or equal to four great Circles; we shall then have the whole

whole Superficies of the Sphere $= 2\,RP = \frac{1}{2}\,RP \times 4 =$ four great Circles; and equal to the whole curved Cylindrick Surface, and the Parts reſpectively equal to the Parts that belong to the ſame Parts of the Axis; alſo the Magnitude of the Sphere $\frac{2}{3}\,RRP = \frac{1}{3}\,RP \times 2\,R$, equal to the Product of $\frac{1}{3}\,R$, a third Part of the common Altitude of all the Priſms, drawn into $2\,RP$ the Aggregate of the Baſes, which is now become the Spherical Surface.

Therefore both the Superficies and Magnitude of the Cylinder circumſcribed to the Sphere, is ſeſqui-alter to the Superficies and Magnitude of the inſcribed Sphere, or as 3 to 2: There becauſe the Proportion is as ſix great Circles $= 3\,PP$ to four great Circles $= 2\,RP$; here becauſe the Proportion is as RRP to $\frac{2}{3}\,RRP$: Which is the very Invention of *Archimedes* ſo much celebrated.

The ſame would be had a little ſhorter, if in the Parallelepiped upon the plain Baſe $2\,RP$ of the Altitude R, compoſed of minute Parallelepipeds, all their Vertices were immediately ſuppoſed to be contracted into one Point C: That the Aggregate of the Baſes continuing as before $= 2\,RP$, thoſe Parallelepipeds may be reduced to ſo many Pyramids, having their Vertices meeting at the Center of the Sphere, whoſe Radius R is the common Altitude of all the Pyramids, and the Spherical Superficies is the Aggregate of all the Baſes. For $\frac{1}{3}\,R$, a third Part of the common Altitude, drawn into $2\,RP$ the Aggregate of the Baſes, exhibits as before the Magnitude of the Sphere $\frac{2}{3}\,RRP$, and the Surface of the Sphere $= 2\,RP$.

In like manner this may be accommodated to the Spherical Sector, by drawing $\frac{1}{3}\,R$, a third Part of the common Altitude of all the Pyramids in it, into a Portion of the Spherical Surface cut off by a Plain: Which is to the whole Spherical Surface, as the Part cut off of the Diameter or Axis is to the whole Diameter; as was ſhewn above.

Now the Reaſon of this whole Proceſs depends on theſe Principles. That a Figure compoſed of Triangles is half the Figure compoſed of Parallelograms, upon the ſame Baſes and of equal Height. That I call a Convolute Figure, and this an Evolute. And that a Figure of Pyramids is a third Part of a Figure of Parallelepipeds, on the ſame Baſes and equal Height. That I call a Complicate Figure, and this an Explicate. Theſe Principles may be accommodated in a thouſand Manners to Curvilinear Figures, whether Superficial or Solid, however perplexed and intricate.

Improvements in England *in the Reſolution of Equations in Numbers, by Mr.* J. Collins. N. 46. *p.* 929. Apr. An. 1669.

XVIII. 1. It hath been obſerved by divers of this Nation, that in any Equation, howſoever affected, if you give a Root, and find the abſolute Number or Reſolvend, (which *Vieta* calls *Homogeneum Comparationis*;) and again give Roots and find more Reſolvends; that if theſe Roots, or rather Rank of Roots, be aſſumed in Arithmetical Progreſſion, the Reſolvends, as to their firſt, ſecond, or third Differences, &c. imitate the Laws of the pure Powers of an Arithmetical Progreſſion, of the ſame Degree, that the higheſt Power, or firſt Term of the Equation is of. *E. g.* In this Equation $aaa - 3aa + 4a = N$.

If

If a be = { 10, 9, 8, 7, 6 } Then N or the Absolutes or Resolvends will be found to be,

a	N	1 diff.	2 diff.	3 diff.
10	740	218		
9	522	170	48	
8	352	128	42	6
7	224	92	36	6
6	132			

To wit, the 3d Differences of thoſe Abſolutes are equal, as in the Cubes of an Arithmetical Progreſſion.

2. To find what Habitude thoſe Differences have to the Coefficient of the Equation, 'tis beſt to begin from an Unit.

3. In any Arithmetical Progreſſion, if you multiply Numbers by Pairs, you ſhall create a Rank of Numbers whoſe ſecond Differences are equal; and if by Ternaries, then the third Differences of thoſe Products ſhall be equal. And how to find the greateſt Product of an Arithmetical Progreſſion of any Number of Terms having any common Difference aſſigned, contained in any Number propoſed, is ſhewed by *Paſcal* in his Tract *Du Triangle Arithmetique*, where he applies it to the Extraction of the Roots of Simple Powers.

4. It appears, how this Rank may be carried eaſily by Addition, till you have a Reſolvend either equal or greater or leſs than that propoſed.

5. When you have a *Majus* and *Minus*, you may interpole as many more Terms in the Arithmetical Progreſſion as you will, that is to ſay, Subdivide the common Difference in the Arithmetical Progreſſion, and render it leſs; and then renew, and find the Reſolvends, which are eaſily obtained out of the Powers and their Coefficients, which are ſuppoſed known, and may be readily raiſed from a Table of Squares and Cubes, *&c*, with which kind the Reader may be furniſhed in *Guildini Centrobaryca*, and *Babington*'s *Fireworks*. By this means you may obtain divers Figures of the Root; and then the General Method of *Vieta* and *Harriot* runs away more eaſily, and is ſo far improved, that after any Figure is placed in the Root, moſt certain Characters are given to know, by Aid of the ſubſequent Dividend and Diviſor, whether the Figure before aſſumed be too great or too ſmall: Or, laſtly, it may be well concluded, that as in Logarithms, when you propoſe ſuch an one as is not abſolutely given in the Canon, you do, by proportional Work, uſing the Aid of their firſt Differences (when their abſolute Numbers differ by Unite) find the abſolute Numbers true to 5 or 6 Places farther than the Canon gives it (the Reaſon whereof is, that the firſt Differences do likewiſe agree to about the ſame Number of Places); that, I ſay, the like may be done in Equations, after divers of the firſt Figures of the Roots are found, provided there be the like Agreement in the firſt Differences of the Interpoled Reſolvends.

Moreover, we ought here to take notice of a more ſubtile kind of Interpolation, common to all Gradual Ranks or Progreſſions of Numbers, wherein Differences happen to be equal: Of which kind the Reader may find Examples in *Briggii Arithmetica Logarithmica*, *&* *Trigonometria Britannica*, relating to Logarithms, Sines, and the Powers of an Arithmetical Progreſſion: but

But the Method there delivered may be rendered more eaſy and general, *viz.* by Aid of a Table of Figurate Numbers, by deriving Generating Differences ſought from thoſe given; a Doctrine that eaſily flows from *Mercator's Logarithmotechnia*, and of uſe in the Caſe in hand, ſhould we ſuppoſe theſe Powers and their Coefficients unknown, or a Table of Squares and Cubes wanting, and give nothing more than a few Reſolvends belonging to equal Moments or Spaces. And this may likewiſe be of good Uſe in *Gauging*, when having the Contents of a Solid, for every three Inches more or leſs given, without knowing the Dimenſions of the Figure, and even in moſt Caſes, when the Differences are progreſſive of one Kind, without knowing the Figure itſelf, having nothing given but its Contents at ſeveral equal parallel Diſtances, each ſuch Diſtance may be ſub-divided, and made as many as you pleaſe, and the reſpective Contents found by this general Method of Interpolation.

After one Root is obtained, the Methods of *Huddenius* and others will depreſs the Equation ſo as to obtain more, and conſequently all of them.

6. It is eaſy by a Table of Figurate Numbers to give the Sum of any ſuch Rank, or any Term in it relating to a known Part of the Series of Equals or Roots; but *è converſo*, giving the Reſolvend to find the Root, comes to an Equation as difficult as that propoſed; as in Dr. *Wallis's* Chapter of *Figurate Numbers.*

7. Some affirm, they can give good Approaches for the obtaining a Root of any pure Power, affected Equation, or for the finding any of the mean Proportionals in any Rank between the two Extremes given.

8. Others pretend to have found out the Method (incited thereto by an Example in *Albert Gerard's Invention Nouvelle en Algebre, à Amſterdam* 1629) ſo much, by comparing of Equations, to increaſe or diminiſh the unknown Root of an Equation, as to render it a whole Number (or leſs differing therefrom than any Error aſſigned) and by *Albert Gerard's* Method of Aliquot Parts to find the ſame, and thereby the Root ſought, altho' it be a mixt Number, Fraction, or Surd.

Probably this may ſympathize with what is promiſed by the learned *Huddenius in Annexis Geometriæ Carteſianæ*, where he ſaith he intended not then to publiſh certain Rules he had ready; whereof one was to find out all the Irrational Roots both of Literal and Numeral Equations: This muſt be underſtood, when ſuch Roots are poſſible; for 'tis certain there are Infinite Equations, whoſe Roots are no ways explicable, either in Whole or Mixed Numbers, Fractions or Surds, and can be no otherwiſe explained, but by a *quamproxime*.

9. The *Author of this Narrative* conſidering that the Conic Sections may be projected from leſſer Circles placed on the Sphere, and thence eaſily (otherwiſe than hitherto hath been handled) deſcribed by Points, and that by their Interſections, ſome Spherick Problem is determined; accordingly, he found that this following Problem, according to the various Situation of the Eye, and of the projecting Plan, would take in all Caſes. *Ibid. p.* 932.

The Distances of an unknown Star are given from two Stars of known Declination and Right Ascension; the Declination and Right Ascension of the unknown Star is required.

And saith, He hath observed, that, admitting the Mechanism of dividing the Periphery of a Circle into any Number of equal Parts, or (which is equivalent) the Use of a Line of Chords, that this Problem, wherever the Eye be placed, may be resolved by plain *Geometry*, and yet the Eye shall be so placed, as to determine it by the Intersection of the Conic Sections; consequently those Points of Intersection (the Species and Position of the Figures being given) may be found without describing any more Points than those sought; and the Lengths and Ordinates falling from thence on the Axes of either Figure calculated by Mixed *Trigonometry*, and hence likewise the Roots of all *Cubick* and *Biquadratic Equations* found by *Trigonometry*.

For giving, from the *Mesolabe* of *Slusius*, the Scheme that finds these Roots, it will then be required to fit those Sections into Cones, which have their Vertex either in the Center, or an assigned Point in the Surface of the Sphere, to which they relate as projected, and proceed to the Resolution of the Problem proposed; and how to fit in those Sections, see the *seven Books of Apollonius*, *Mydorgius*, the 3d Volume of *Des Cartes*'s Letters, *Leotaudi Geometria Practica*, *Andersonii Exercitat. Geometricæ*.

As to the Problem itself, it is determined on the Sphere by the Intersections of the two lesser Circles of Distance, whose Poles are the known Stars. And this Problem hath divers Geometrick Ways of Resolution.

1. By plain Geometry (in the Sense before mentioned): Supposing a Plain to touch the Sphere or the North Pole; if the Eye be at the South Pole, projecting those Circles into the said Plane, they are still Circles (by reason of the sub-contrary Sections of the visual Cones) whose Centers fall in the Sides of the Right-lined Angle, made by the projected Meridians, that pass thro' the known Stars; and thus the Problem is easily solved in this Manner.

2. If it be required to be performed by Conic Geometry; In one Case it may be done, by placing the Eye at the Center of the Sphere, and projecting as before; to wit, when the longer Axes of the Figures being produced, concur above the Vertex: Here the Problem is determined by the Intersections of two Conic Sections (whereof a Circle cannot be one, unless its Center be in the Axis of the other Figure): And in this second Case, these Points of Intersection fall in the same Right Line or projected Meridian they did before, but at a more remote Distance from the Pole Point, to wit, in the former Supposition the Polar Distance was measured by a Right Line, that was the double Tangent of half the Arch; here it is the Tangent of the whole Arch. Hence it is evident how one Projection may beget another, yea, infinite others, altering the Scale, and how the lesser Circles in the *Stereographick* Projection help to describe the Conic Sections in the *Gnomonick* Projection: But (to reduce the Matter to the common Radius) if we suppose two Spheres equal, and so placed about the same Axis, that the Pole Point of the one shall pass thro' the Center of the other, and the Touch-Plain to pass thro' the said Center or Pole Point; and that a lesser Circle hath the

same

ſame Poſition in the one as in the other: then, if the Eye be at the South Pole of the one, it is at the Center of the other; and any projected Meridian drawn from the projected Pole Point to paſs thro' both the Projections of theſe leſſer Circles, the Diſtances of the Points of Interſection are the Tangents of the half and the whole Arch of the Meridian ſo interſected. But as to the Points of Interſection, which determine the Problem propoſed, they may be found without the Aid of the former Way, from a *Gnomonick* and *Stereographick* Method of meaſuring and ſetting off the Sides and Angles of Spherical Triangles in thoſe Projections, which is neceſſary in what follows.

3. If the Problem is to be performed by mixed Geometry, as by a Circle, and either a Parabola, Hyperbola, or Ellipſis, the Circle may be conceived to be the ſub-contrary Section of a Cone projected by the Eye at the South Pole, and any of the reſt of the Sections by the Eye at the Center of the Sphere.

4. If by any of the Conick Sections however poſited, the projecting Plane may remain the ſame; but the Eye muſt be in ſome other Part of the Surface of the Sphere, and not in the Axis.

The Conſtruction of Cubick and Biquadratic Equations, by a Parabola and Circle. By Mr. Edm. Halley. N. 188. *p.* 335. Jul. An. 1687. *Fig.* 48.

XIX. 1. The Conſtruction delivered by *Des Cartes*, which eaſily finds the Roots of all Cubic and Biquadratic Equations, wherein the ſecond Term is wanting, I may here ſuppoſe as known: Yet ſince the following depends upon it, I ſhall here ſubjoin the Rule taken from his Geometry, changing a few Things (as I hope) for the better.

When the ſecond Term is wanting, all Cubic Equations may be reduced to this Form, $z^3 * apz : aaq = 0$. And all Biquadratics to this, $z^4 * apz^2 : aaqz : a^3r = 0$. Where a denotes the *latus rectum* of any given Parabola, which may be uſed in the Conſtruction; or making $a = 1$, the Equations will be $z^3 * pz : q = 0$, and $z^4 * pz^2 : qz : r = 0$.

Now the Parabola F A G being given, whoſe Axis is A C D K L, and its *latus rectum* a or Unity, let A C be its half, and let it always be placed from the Vertex A within the Figure. Then take $CD = \frac{1}{2}p$, in the Line A C continued towards C, if in the Equation it is $-p$, or the other Way if it ſhall be $+p$. Then from the Point D, or from C if $p = 0$, the Perpendicular to the Axis $DE = \frac{1}{2}q$ is to be raiſed, to the Right Hand if it ſhall be $-q$, but on the other Side of the Axis if it ſhall be $+q$. Then a Circle deſcribed with Center E, and Radius A E, if the Equation be only Cubic, will cut the Parabola in ſo many Points F and G, as it has true Roots, of which the Affirmative, as G K, will be to the Right Side of the Axis, and the Negative, as F L, will be on the Left Side.

But if the Equation is Biquadratic, the Radius of the Circle A E muſt be increaſed or diminiſhed, by adding if it is $-r$, or ſubtracting if it is $+r$, the Rectangle ar from its Square, or the Plain contained by the *latus rectum* and the given Quantity r; which is eaſily done Geometrically. Now the Interſections of this Circle with the Parabola will exhibit all the true Roots of the Biquadratic Equation, by letting fall Perpendiculars to the Axis;

the

the Affirmatives to the Right Side of the Axis, and the Negatives to the Left Side. But I leave to *Des Cartes*, its Inventor, the Demonſtration of the whole.

Here it is to be obſerved, that I endeavour always to have the Affirmative Roots on the Right Side of the Axis, to avoid that Confuſion that will neceſſarily ariſe, from a Multitude of Cautions and Exceptions, the Reaſon of which would by no means be evident.

Theſe Things being premiſed, that we may come to the Conſtruction of ſuch Equations alſo, where the ſecond Term is preſent; the Rule for taking away the ſecond Term muſt be conſidered, and for reducing the Equation to another, which may be conſtructed by the foregoing Method. Now all the Cubic Equations of this Claſs may be reduced to this Form, $z^3 : bz^2 : apz : a^2q = 0$, or to this, $z^3 : bz^2 : * \, a^2q = 0$. And the Biquadraticks to this $z^4 : bz^3 : apz^2 : a^2qz : a^3r = 0$, or to this $z^4 : bz^3 : * : a^2qz : a^3r = 0$, or to this, $z^4 : bz^3 : apz^2 : * : a^3r = 0$, or laſtly to this, $z^4 : bz^3 : * : * : a^3r = 0$. Out of all theſe, as they may be differently connected by the Signs $+$ and $-$, there ariſes a vaſt Variety. Whence a general Rule comprehending them all muſt needs be very difficult and obſcure, unleſs it be treated in the following Manner, and ſo delivered from its Perplexities.

In Biquadratics the ſecond Term is taken away, by making $x = z + \frac{1}{4}b$, if it be $+b$ in the Equation; or $x = z - \frac{1}{4}b$, if it be $-b$. Hence in the firſt Caſe $x - \frac{1}{4}b$, and in the ſecond $x + \frac{1}{4}b$, is equal to z; and in any Equation propoſed inſtead of z ſubſtituting its equal Quantity, there ariſes a new Equation in which the ſecond Term is wanting, all whoſe Roots x either exceed or are deficient from the Root ſought z, by a given Difference $\frac{1}{4}b$.

Example 1. $z^4 + bz^3 - apz^2 - a^2qz + a^3r = 0$. Make $x - \frac{1}{4}b = z$, then $x^2 - \frac{1}{2}bx + \frac{1}{16}b^2 = z^2$, $x^3 - \frac{3}{4}x^2b + \frac{3}{16}xb^2 - \frac{1}{64}b^3 = z^3$, and $x^4 - bx^3 + \frac{3}{8}b^2x^2 - \frac{1}{16}b^3x + \frac{1}{256}b^4 = z^4$. Hence by Subſtitution

$$\begin{aligned}
x^4 - bx^3 + \tfrac{3}{8}b^2x^2 - \tfrac{1}{16}b^3x + \tfrac{1}{256}b^4 &= +z^4 \\
+bx^3 - \tfrac{3}{4}b^2x^2 + \tfrac{3}{16}b^3x - \tfrac{1}{64}b^4 &= +bz^3 \\
-apx^2 + \tfrac{1}{2}abpx - \tfrac{1}{16}apb^2 &= -apz^2 \\
-a^2qx + \tfrac{1}{4}a^2bq &= -apqz \\
+a^3r &= +a^3r
\end{aligned}$$

The Sum of all theſe becomes a new Equation, in which the ſecond Term is wanting, and which therefore may be conſtructed by *Cartes*'s Rule; putting inſtead of $\frac{1}{2}p$ half the Coefficient of the third Term divided by a or the *la-*

tus

tus rectum, that is, $-\frac{3bb}{16a}-\frac{1}{2}p$; and instead of $\frac{1}{2}q$ half the Coefficient of the fourth divided by aa, or $+\frac{b^3}{16a^2}+\frac{pb}{4a}-\frac{1}{2}q$. The Parts affected with the Sign + are to be placed on the Left Side of the Axis, and those affected with — on the Right Side, that the Center of the Circle may be had, as is requisite for the Construction; so that the Intersections of this with the Parabola, by Perpendiculars let fall upon the Axis, may exhibit all the true Roots x, the Affirmatives on the Right Side of the Axis, and the Negatives on the Left Side. For since $x-\frac{1}{4}b=z$, by drawing a Line parallel to the Axis, on its Right Side and at the Distance $\frac{1}{4}b$, those Perpendiculars terminated at this Parallel will exhibit all the Roots z, the Affirmatives to the Right Hand and the Negatives to the Left. As for the Radius of the Circle, it will be had by adding the negative Parts, and subtracting the affirmative Parts of the fifth Term divided by aa, from the Square of the Line A E drawn from the Center E (when found) to the Vertex of the Parabola A. And this may generally be done by taking instead of the Line A E the Line A O, which is terminated at O the Intersection of the Parabola and the aforesaid Parallel. For its Square contains (as may easily be proved) all the Parts of the fifth Term accruing to the new Equation, by taking away the second Term: And it only remains that the Square of E O may be increased, if there is $-r$ in the Equation, or diminished if there is $+r$, by the Addition or Subtraction of the Rectangle ar; whence the Square of the Radius of the Circle required is composed.

This is Mr. *Baker*'s Method of finding the Central Rule, but free from all his Cautions, and easy enough. The only Difference arises from hence, that I determine the Center of the same Circle by the Axis, and he by the Parallel to the Axis; and that I always find the affirmative Roots on the Right Side of the Axis, which he places sometimes on the Right and sometimes on the Left.

As for what belongs to Cubic Equations, they must be reduced to Biquadratics, before they can be constructed by the same general Rule. And this is done by multiplying the Equation proposed by its Root z, whence a Biquadratic Equation arises, in which the last Term, or r, is wanting. Wherefore taking away the second Term, and the Center E being found, the Line E O is the Radius of the Circle; since $ar=0$, and in the new Equation the whole fifth Term arises, by only taking away the second Term.

Example 2.] $z^3-bz^2+apz+aaq=0$. This multiplied by z becomes $z^4-bz^3+apz^2+aaqz=0$. To take away the second Term make $x+\frac{1}{4}b=z$.

$$\text{Then } x^4+bx^3+\frac{3}{8}b^2x^2+\frac{1}{16}b^3x+\frac{1}{256}b^4 = z^4$$

$$-bx^3-\frac{3}{4}b^2x^2-\frac{3}{16}b^3x-\frac{1}{64}b^4 = -bz^3$$

$$+apx^2$$

$$+ \quad apx^2 + \frac{1}{2}abpx + \frac{1}{16}apb^2 = +apz^2$$
$$+ \quad a^2qx + \frac{1}{4}a^2bq = +a^2qz$$

In this new Equation, half the Coefficient of the third Term divided by a, that is $-\frac{3bb}{16a} + \frac{1}{2}p$, is to be used instead of $\frac{1}{2}p$; and half the Coefficient of the fourth Term, divided by aa or the Square of the *latus rectum*, that is $-\frac{b^3}{16a^2} + \frac{pb}{4a} + \frac{1}{2}q$, is to supply the Office of $\frac{1}{2}q$ in *Cartes*'s Construction. Whence the Center E is determined. Then a Parallel being drawn to the Axis at the Distance of $\frac{1}{4}b$, on its Left Side, because of $x + \frac{1}{4}b = z$, and let the Intersection of this with the Parabola be O. A Circle described with Center E, and Radius E O, will cut or touch the Parabola in so many Points at the Equation has true Roots; which Roots or Values of z are exhibited by Perpendiculars let fall from those Points upon the Parallel to the Axis; the Affirmatives to the Right Hand, and the Negatives to the Left.

If in the Equation either the third or fourth Term, or both shall be wanting, there will be no Difference to be observed in investigating the Central Rule, but the Quantities p or q vanishing, those Parts of the Lines C D and D E arising from those Quantities will be absent; so that we must proceed with the remaining Coefficients of the third and fourth Terms in the new Equation, as is prescribed in the foregoing Examples.

Hitherto we have pursued *Baker*'s general Method, than which we can expect none that will be easier or readier; whether a Parabola is assumed for the Construction, or any other Curve, and when the Equation ascends to the Biquadratic. And whilst I am writing this there occurs to me a Geometrical Effection of the Central Rule, which is expeditious beyond Hope, and will abundantly satisfy the Curious in these Matters.

Fig. 49. The Parabola N A M being described, whose Vertex is A, its Axis A B C, and *latus rectum* a, let the Equation be reduced to this Form, $z^4 : bz^3 : apz^2 : a^2qz : a^3r = 0$; or to this, $z^3 : bz^2 : apz : a^2q = 0$, if it be a Cubic Equation. Then at the Distance of B D $= \frac{1}{4}b$ let the Line D H be drawn parallel to the Axis, to the Left Hand if it be $-b$, or to the Right if $+b$, meeting the Parabola in the Point D, from whence let fall B D perpendicular to the Axis. In the Line A B continued towards B make B K $= \frac{1}{2}a$, and draw the Line D K both ways indefinitely. Then make K C $= 2$ A B in the Axis always continued beyond K; and if the Quantity p is affected with the Sign $-$, take C E $= \frac{1}{2}p$ also towards the same Side, but contrariwise if it is $+p$; and from the Point E let there be raised E F perpendicular to the Axis, or from the Point C if the Quantity p is absent, meeting in F the Line D R produced if need be; which Point F is the Center of the Circle required if q be absent, but if it be present in F E, continued if need be, must be taken a Line F G $= \frac{1}{2}q$, to the Left if it is $+q$, but to to the Right if $- q$. And the Point G will be the Center of the Circle proper for the propofed Conftruction, and its Radius will be the Line G D, if r is wanting or if the Equation is only Cubical. In the Biquadratics the Square of this muft be increafed or leffened, according as it is $- r$ or $+ r$, by the Rectangle of r and the *latus rectum*. The Circle being thus defcribed, a Perpendicular being let fall upon the Line D H from its Interfections with the Parabola, thofe that are on the Left Hand, as N O, will always reprefent the negative Roots of the Equation, and thofe on the Right Hand, as M L, the affirmative Roots.

Cubic Equations are conftructed otherwife, and fomething more fimply; by a Rule of *Van Schootens*, by which alfo the Roots are referred to the Axis. But becaufe the Inventor himfelf does not deliver his Manner of finding it, nor the Demonftration of his Invention, it may not be amifs to fhew here the Foundation of the fame, and to make the Geometrical Effection fomething more neat, and to free it from fome Cautions in which it is involved.

This Rule is derived from hence, that every Cubic Equation may be reduced to a Biquadratic, in which the fecond Term is wanting. This is done by multiplying the Equation propofed into $z - b = 0$, if there is $+ b$ in the Equation; or into $z + b = 0$, if it be $- b$. Then the new Equation fo produced will have the fame Roots with the Cubic, and another befides equal to $- b$, if there is $- b$ in the Equation; or on the contrary.

Suppofe we were to conftruct $z^3 - z^2 b + a p z + a a q = 0$. Multiply this by $z + b$; it becomes

$$\left.\begin{array}{l} z^4 - b z^3 + a p z^2 + a^2 q z \\ \quad + b z^3 - b^2 z^2 + a b p z + a^2 b q \end{array}\right\} = 0.$$

Here the fecond Term is wanting, and the Coefficient of the third $- b b + a p$ gives $- \frac{b b}{2 a} + \frac{1}{2} p$ to be fubftituted for $\frac{1}{2} p$ or C D in *Cartes*'s Conftruction; and of half the Coefficient of the fourth Term is made $+ \frac{1}{2} q + \frac{b p}{2 a}$, to be ufed inftead of $\frac{1}{2} q$ or D E, and fo the Center of the Circle fought is determined. Then becaufe one of the Roots of the new Equation is given, which is $-$ or $+ b$, a Point alfo in the Circumference will be given, and therefore the Radius. Laftly, the Circle being defcribed, Perpendiculars let fall upon the Axis from its Interfections with the Parabola will exhibit the Roots of the Equation, the affirmative and negative as mentioned before.

Now the Center of the Circle is found by a very eafy Conftruction, which is preferable to all others in Cubic Equations. Let A be the Vertex and A F the Axis of the defcribed Parabola A M D. At a Diftance equal to b let D K be drawn parallel to the Axis, at the Right or Left Hand according as it is $+ b$ or $- b$, which meets the Parabola in the Point D. With Centers D and A, and with equal Radius's, let occult Arches be defcribed interfecting one another on both Sides, and through the Points of Interfection Fig. 50.

draw

draw the indefinite Right Line P C, which may be perpendicular to the supposed Line A D at its middle, and meet the Axis at the Point E. From E, either below or above according as there is either $-p$ or $+p$ in the Equation, make $EF = \frac{1}{2}p$; and from F, or from E if p is wanting, draw the Perpendicular F G meeting the Line B C in the Point G; and in G F produced make $GH = \frac{1}{2}q$, to the Right if there is $-q$ in the Equation, or otherwise to the Left. Then will H be the Center required, and H D the Radius of the Circle; which will shew all the Roots as before, by means of Perpendiculars let fall from its Intersections with the Parabola to its Axis.

The Number of Roots in such Equations, with their Limits and Signs; by Mr. Edm. Halley. N. 190. p. 389. Nov. An. 1687.

It appears from *Des Cartes* and from what has been already said, as well in Cubics as in Biquadratics, that the Roots may be represented by letting fall Perpendiculars upon the Axis, or a given Diameter of the given Parabola, from the Intersections of that Curve with a Circle. And since a Circle cutting a Parabola must cut it either in four or in two Points, it is plain that in Biquadraticks there must necessarily be two or four true Roots, either affirmative or negative; as also if perhaps the Circle may touch it, in which we may infer an Equality of two Roots of the same Sign. But in Cubicks, because one of the Intersections is required for the Construction, the one or the three remaining represent the one or the three Roots, as in the Case of Contact, whence it is plain that there are found two equal Roots, and that the Problem from whence the Equation results is truly a plane Problem.

Therefore all Cubic Equations however affected are explicable with one or with three Roots, as always possible; admitting negative Roots for true ones. So Biquadraticks, of which the last Term r is affected with the Sign

Fig. 4.

$-$, are explicable by two or four. But if there is $+r$ in the Equation, and it is so great as that $\sqrt{GDq - ar}$ is less than that a Circle can reach the Parabala in any Point, when described with that Radius and the Center G; the given Equation is totally impossible, nor is it explicable by any Root affirmative or negative. But of this more afterwards.

Now whereas there is so great a Difference between the Cases of Cubick and Biquadratick Equations, as that they cannot be comprehended together; I shall first treat of Cubicks, and then of the other. Now Cubicks may be constructed by an infinite Number of Circles in a given Parabola, but Biquadraticks with one only, at least in this Method. And that because by putting $z - e$, or any Indeterminate, equal to nothing, the Cubick Equation is reduced to a Biquadratick, having the same Roots with the Cubick, and another besides wh.ch is equal to e. Whence it is that the Cubick may be constructed by so many different Circles as there are different Values of e, that is infinite. But among these Constructions that is by far the most easy, which I gave above in the last Paragraph. Yet there is another not much inferior to it, which seems more accommodated to the Discovery of the Number of the Roots and their Limits, and which derives its Original from taking away the second Term, by supposing (as is commonly done) $x = z +$ or $-$ the third Part of the Coefficient of the second Term. Here it follows:

Fig. 51.

The Parabola A B Y being given, its Vertex A, its Axis A E, and its *latus rectum a*; let the Equation be reduced to the usual Form $z^3 : bz^2 :$

$apz:$

$apz : a^2 q = 0$. Then at the Diſtance $\frac{1}{2} b$ let B K be drawn parallel to the Axis, to the Right Hand if it be $+ b$, or elſe to the Left Hand, meeting the Parabola in B. And let an indefinite Perpendicular be erected D P on the middle of the ſuppoſed Line A B, meeting the Axis in the Point G. From B let fall B C perpendicular to the Axis, and let G E be always made equal to A C with its Direction downwards. From E make $EH = \frac{1}{2} p$ with its Direction upwards if it be $+ p$ in the Equation, but otherwiſe downwards, and from the Point H (or from E if p be wanting) let the Perpendicular H Q be raiſed meeting the indefinite Line D P in the Point O. Laſtly in the indefinite Line H Q make $OR = \frac{1}{2} q$, to the Right from O if it be $- q$, to the Left if $+ q$. Then a Circle deſcribed with Center R, and Radius R A, will cut the Parabola in ſo many Points as the Equation propoſed has true Roots, and theſe will be the Perpendiculars Z Y let fall from the Points of Interſection Y upon B K parallel to the Axis; of which thoſe on the Right Side of B K will be the affirmative Roots, and thoſe on the Left Side the Negative.

The Convenience of this Conſtruction conſiſts in this, becauſe it is performed by a Circle paſſing through the Vertex, juſt as if the ſecond Term were wanting. Therefore for determining the Number of Roots, it is enough to know the Properties of the Locus or the Curve-Line which diſtinguiſhes the Spaces, where if the Center of the Circle be placed that paſſes through the Vertex of the Parabola, its Circumference will cut it either in one or in three Points: That is, to define the Nature of the Curve, upon which ſhall fall the Centers of all the Circles, that paſs through the Vertex and then touch the Parabola.

But that Locus is the Paraboloid, which with the great *Wallis* we will call Semicubical, or in which the Cubes of the Ordinates are to one another as the Squares of the Portions of the Axis reſpectively. Its *latus rectum* is $\frac{27}{8}$ of the *latus rectum* of the given Parabola, and its Vertex is the Point V, A V being half the *latus rectum* of the ſame Parabola. That is, if Unity be aſſigned for the *latus rectum* of the given Parabola, $\frac{8}{27}$ of the Cube of the Ordinate will be equal to the Square of the Part of the Diameter, or the Cube of $\frac{2}{3}$ V H will be equal to the Square of H R, if R be the Center of the Circle that paſſes through the Vertex of the Parabola, and afterwards touches it. This is that Curve which firſt of all Mortals our Mr. *Neil* demonſtrated to be equal to a Right Line, who on this Account has been celebrated among the chief of Geometricians. And its Properties have been inquired into by the learned Dr. *Wallis*, at the End of his Book concerning the Ciſſoid, and *Huygens*, *Prop.* 8, 9. of the Evolution of Curve Lines, and other ingenious Men whoſe Writings the Reader may conſult. This Curve being deſcribed on each Side of the Axis of the Parabola, that is, V N L, and V P X, will comprehend a Space, in which if the Center of the Circle be put, which paſſes through the Vertex A, that Circle will cut the Parabola in three other Points. But the Spaces that are more remote from the Axis will ſupply the Centers of Circles which will cut the Parabola but in one Point beſides the Vertex.

Theſe Things being rightly underſtood, we ſhall now proceed to determine the Number of the Roots. And firſt let the ſecond Term be wanting, and let the *latus rectum* be Unity, or $AV = \frac{1}{2}$. In the Conſtruction VH is $\frac{1}{2}p$, and $HR \frac{1}{2}q$. And as $\frac{1}{2}p$ is to be put upwards from V if it be $+p$, the Center of the Circle is always to be placed without the Space LVX, and therefore the Equation is explicable by one Root only, affirmative if it be $-q$, negative if $+q$, which Roots are inveſtigated by *Cardan's* Rules. But if it is $-p$, then $VH = \frac{1}{2}p$ muſt be taken below, and it may happen that HR may fall between the Axis and the Curve VX or VL, namely if the Cube of 2 VH or $\frac{1}{3}p$ is greater than the Square of $\frac{1}{2}q$, or $\frac{1}{27}p^3$ greater than $\frac{1}{4}qq$, in which Caſe there are three Roots, two negative if it is $-q$, and one Affirmative equal to their Sum; or if it is $+q$, then two affirmative and one negative. But if $\frac{1}{27}p^3$ is leſs than $\frac{1}{4}qq$, then only one Root is found, affirmative if it is $-q$, but negative if $+q$. And theſe Things are commonly taught by thoſe who treat of this Part of Geometry.

Now let all the Terms be preſent, and firſt let this Equation be propoſed $z^3 - bz^2 + pz - q = 0$, to which we have adapted Figure 51. In the Conſtruction of this, $BC = \frac{1}{3}b$, $VG = \frac{1}{2}AC = \frac{1}{18}bb$, $VE = \frac{1}{6}bb$, $VH = \frac{1}{6}bb - \frac{1}{2}p$, $GH = \frac{1}{9}bb - \frac{1}{2}p$, or $\frac{1}{2}p - \frac{1}{9}bb$, hence $HO = \frac{1}{27}b^3 - \frac{1}{6}bp$, or $\frac{1}{6}bp - \frac{1}{27}b^3$, and HR, or the Diſtance of the Center of the Circle R from the Axis, is always the Difference between $\frac{1}{6}bp$ and $\frac{1}{27}b^3 + \frac{1}{2}q$; which if they are equal, the Center falls in the Axis; if $\frac{1}{6}bp$ is greater than $\frac{1}{27}b^3 + \frac{1}{2}q$, it falls to the Left of the Axis, but if leſs, to the Right. If the Square-root of the Cube of $\frac{2}{3}VH$, (that is, of $\frac{1}{9}bb - \frac{1}{3}p$, which we may call d,) that is, if $\sqrt{ddd}$ is greater than HR, or the Difference between $\frac{1}{27}b^3 + \frac{1}{2}q$ and $\frac{1}{6}bp$, the Center R is found within the Space HPV, which is circumſcribed by the Paraboloids VPX and VNL, and the indefinite Right Line DNP. And therefore the Circle will cut the Parabola in three Points Y, Y, Y, ſituate to the Right Hand of the Line BK, and therefore the Equation has three affirmative

Roots.

Roots. But the Center being without this Space, N U P, it can be explained but by one affirmative Root. Here we may take Notice, that the Right Line D P that touches the Paraboloid V P X in the Point P, E P being $\frac{1}{27} b^3$, will cut the other V N L in the Point N, ſo that letting fall the Perpendicular N F upon the Axis, V F will be a fourth Part of E V, or $\frac{1}{24} bb$, but N F will be $\frac{1}{108} b^3$. But V W, which from the Point V being drawn perpendicular to the Axis, and meets the Line D P in W, is equal to $\frac{1}{54} b^3$, or $\frac{1}{2}$ E P.

Hence we may ſafely conclude, if in the Equation p is greater than $\frac{1}{3} bb$, or q greater than $\frac{1}{27} b^3$, there is only one Root to be had, and that affirmative. Therefore *Des Cartes*'s Rule is not true, (*pag.* 70. Edit. *Amſt.* 1659) wherein he aſſerts, that ſo many true Roots are to be had as in the Equation there are Changes of the Signs + and —; nor is it to any Purpoſe that *Schooten* in his Commentaries excuſes this Miſtake. For infinitely more Equations may be contrived of the foregoing Form, having three Changes of the Signs, which have but one Root, than which have three Roots. Alſo the fifth Propoſition of the fifth Section of our *Harriot's Ars Analytica*; as likewiſe the 18th Problem of *Vieta's Numeroſa Poteſtatum Reſolutio*, are not ſufficiently found, ſince from the Limitations they have there given, that ſhould belong to the whole Parallelogram P I V W, which we have now proved to belong only to the Space N V P; that is, that it ſhould afford a Center for a Circle, which cuts the Parabola in three other Points beſides the Vertex.

Now the Quantity q, or the laſt Term, when b and p are given in ſuch Conditions that p is leſs than $\frac{1}{3} bb$, is accurately limited by the foregoing Equation $\sqrt{ddd} = \frac{1}{27} b^3 + \frac{1}{2} q \backsim \frac{1}{6} bp$, when the Circle touches the Parabola. Therefore $\frac{1}{2} q$ ought to be leſs than $\frac{1}{6} bp - \frac{1}{27} b^3 + \sqrt{ddd}$. But if p is greater than $\frac{1}{4} bb$, it is neceſſary that $\frac{1}{2} q$ ſhould be greater than $\frac{1}{6} bp - \frac{1}{27} b^3 - \sqrt{d^3}$, leſt the Center ſhould fall within the little Space N V W. And with theſe Conditions the Equation is always explicable by three Roots, but otherwiſe by one only. But whether three or one, they are always affirmative, becauſe of the Poſition of the Center R to the Right Hand of the Line D P.

L 2 And

And this is the moſt difficult Caſe, ſo that he that well underſtands what goes before will eaſily underſtand the reſt. Now let be given the Equation $z^3 - bz^2 + pz + q = 0$. Here that three Roots may be had, it is neceſſary that the Center of the Circle may be ſomewhere within the Space P N Δ, included by the Right Lines P N, P Δ, and by the Curve of the Paraboloid N Δ. Wherefore ſince E F $= \frac{1}{8} bb$, p muſt be leſs than $\frac{1}{4} bb$. Now for the Determination of the Quantity q, making $d = \frac{1}{9} bb - \frac{1}{3} p$ as before, then $\sqrt{ddd + \frac{1}{27} bbb - \frac{1}{6} bp}$ muſt always be greater than $\frac{1}{2} q$, that the Center of the Circle may be found in the aforeſaid Space P N Δ: Which when it happens, ſuch an Equation has two affirmative Roots and one negative. But if p is greater than $\frac{1}{3} bb$, or $\frac{1}{2} q$ greater than $\sqrt{d^3 + \frac{1}{27} b^3 - \frac{1}{6} bp}$, the Equation is explicable but by one Root, and that will be negative.

Now let be propoſed the Equation $z^3 - bz^2 - pz - q = 0$. That this Equation may have three Roots, the Center of the Circle muſt be found ſomewhere in the indefinite Space, between the Right Line D P D and the Curve of the Paraboloid P X. Hence the Quantity p will be ſubject to no Limitations, but $\frac{1}{2} q$ muſt always be leſs than $\sqrt{d^3 - \frac{1}{27} b^3 - \frac{1}{6} bp}$ making $d = \frac{1}{9} bb + \frac{1}{3} p$. By this means there will be two negative Roots, and one affirmative; but otherwiſe if $\frac{1}{2} q$ is greater than $\sqrt{d^3 - \frac{1}{27} b^3 - \frac{1}{6} bp}$, one only affirmative Root will be exhibited.

In the fourth Place let the Equation be $z^3 - bz^2 - pz + q = 0$, which has two affirmative Roots and one negative, if the Center of the Circle is found in the indefinite Space between the Right Lines P Δ, P D, and the Curve of the Paraboloid Δ L; that is, ſuppoſing $d = \frac{1}{9} bb + \frac{1}{3} p$, if $\frac{1}{2} q$ is leſs than $\sqrt{d^3 + \frac{1}{27} b^3 + \frac{1}{6} bp}$; but if $\frac{1}{2} q$ is greater than this Quantity, there is only one negative Root.

Now the four remaining Equations, in which there is $+ b$, as to the Limitation of the Number of Roots, do not differ from the foregoing, if the Sign of the laſt Term be changed, keeping the Sign of the third Term; but the Roots that in thoſe were affirmative will here be negative, and *vice verſa*. Thus in the Equation $z^3 - bz^2 + pz - q = 0$, either one or three are negative under the ſame Conditions, but none affirmative. So in

$z^3 + b$

$z^3 + bz^2 + pz - q = 0$, two are negative and one affirmative, if p is lefs than $\frac{1}{3}bb$, and $\frac{1}{2}q$ lefs than $\sqrt{d^3 + \frac{1}{27}b^3 - \frac{1}{6}bp}$, as in $z^3 - bz^2 + pz + q = 0$, two will be affirmative and one negative. But when p or q depart from the prefcribed Limits, here is only one affirmative Root which there was negative. In like manner in $z^3 + bz^2 - pz + q = 0$, either there are two affirmative and one negative Root, or one negative only. Laftly for the fame Reafon in the Equation $z^3 + bz^2 - pz - q = 0$, there are two negative and one affirmative Root, or one affirmative only; as in the Equation $z^3 - bz^2 - pz + q = 0$, there are two affirmative and one negative, or only one Negative, according as $\frac{1}{2}q$ is greater or lefs than $\sqrt{d^3 + \frac{1}{27}b^3 + \frac{1}{6}bp}$.

If the third Term is wanting, or $p = 0$, the Center R always falls in the Line I P E Δ; wherefore if it is $z^3 - bz^2 * - q = 0$, or $z^3 + bz^2 * + q = 0$, there can be only one Root, if 'tis $-b$ the Root is affirmative, if $+b$ negative. But if the Equation is $z^3 - bz^2 * + q = 0$, or $z^3 + bz^2 * - q = 0$, there may be two affirmative and one negative in the former, or one affirmative and two negative in the latter, the Center falling in the Line P Δ, between P and Δ, that is, if $\frac{1}{4}q$ is lefs than $\frac{1}{27}b^3$: But if it be greater, then can be only one Negative in the former, or one Affirmative in the latter.

Hitherto we have fully explained the Number of Roots in Cubic Equations; now we muft add fomething concerning the Quantity of thofe Roots. Here it is firft to be obferved, that every Equation having three Roots may be refolved expeditioufly enough by means of the Table of Sines, that is, by the Trifection of an Angle. For by making $\sqrt{\frac{4}{9}bb - \frac{4}{3}p}$, or $\sqrt{4d}$ if it is $+p$ in the Equation, or $\sqrt{\frac{4}{9}bb + \frac{4}{3}p}$, if it is $-p$, the Radius of the Circle; and making that the Angle to be trifected, whofe Sine is $\frac{\frac{1}{27}b^3 + \frac{1}{6}bp + \frac{1}{2}q}{\sqrt{d^3}}$ in the Table of Sines; when this Angle is found the Sine of a third Part of it, alfo the Sine of the third Part of its Complement to a Semicircle, and their Sum, will be known by the Table of Sines. But thefe Sines are to be drawn into the Radius $\sqrt{\frac{4}{9}bb + \frac{4}{3}p}$, and we fhall have the Quantities (y &, y &, y &, in the Figure) the Sum or Difference of which, and of $\frac{1}{3}b$, as the Cafe requires, will exhibit the true Roots of the Equation. All thefe Things are derived from what has been found by

Cartesius. Now that I may comprehend all the Cases in as short a Compass as possible. I say that in the first Form of Equations, if the Center R falls in the Space V G P, the two Sections Y, Y, fall between A and B, and therefore each of the lesser Roots is less than $\frac{1}{3}b$, but the third and greater always exceeds $\frac{1}{3}b$, but is exceeded by b. Now if it falls in the Space G N V, two are greater than $\frac{1}{3}b$, but less than $\frac{2}{3}b$, and the third is b substracting the two others, and therefore less than $\frac{1}{3}b$. But admitting a Limitation of the Quantity p, the Roots are included within narrower Bounds. For the greatest Root is less than $\sqrt{\frac{4}{9}bb - \frac{4}{3}p} + \frac{1}{3}b$, and greater than $\sqrt{\frac{1}{4}bb - p} + \frac{1}{2}b$. But when $\frac{1}{4}bb$ is less than p, that Limit becomes $\sqrt{\frac{1}{9}bb - \frac{1}{3}p} + \frac{1}{3}b$. The middle Root is always less than $\sqrt{\frac{1}{4}bb - p} + \frac{1}{2}b$, but greater than $\frac{1}{3}b - \sqrt{\frac{1}{9}bb - \frac{1}{3}p}$. The greatest Root never exceeds this Limit, but vanishes with the Quantity q.

In the second Form by the Rules prescribed there are two affirmative and one negative Root, and the Center falling in the Space G P E, one of the Affirmatives is greater and the other less than $\frac{1}{3}b$. But the greater does not exceed b, and the Negative cannot be greater than $\sqrt{\frac{1}{3}bb} - \frac{1}{3}b$, and it is the Difference of b and of the Sum of the Affirmatives. But when the Center is placed in the Space E N G Δ, either of the Affirmatives is greater than $\frac{1}{3}b$, but less than $\sqrt{\frac{1}{3}bb} + \frac{1}{3}b$. But the Negative is always less than $\frac{1}{3}b$. But from p being given, the Limits become nearer, $\sqrt{\frac{1}{4}bb - p} + \frac{1}{2}b$ of the greatest affirmative Root, than which it is always less, and greater than $\sqrt{\frac{1}{9}bb - \frac{1}{3}p} + \frac{1}{3}b$. Yet the other affirmative Root, which diminishes with the Quantity q, is less than this Limit. But the negative Root is always less than $\sqrt{\frac{4}{9}bb - \frac{4}{3}p} - \frac{1}{3}b$, and vanishes when the Quantity q is absent.

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In the third Form are two negative and one affirmative Root; in this, as alſo in the fourth, the Roots are not limited by the Quantity b. Now the Affirmative is always leſs than $\sqrt{\frac{4}{9}bb + \frac{4}{3}p} + \frac{1}{3}b$, but greater than $\sqrt{p + \frac{1}{4}bb} + \frac{1}{2}b$. But the greateſt of the Negatives is always greater than $\sqrt{\frac{1}{9}bb + \frac{1}{3}p} - \frac{1}{3}p$, and leſs than $\sqrt{p + \frac{1}{4}bb} - \frac{1}{2}b$. But the leſſer of the Negatives always is diminiſhed when the Quantity q is diminiſhed.

In the fourth Form the Center falls within the Space L Δ P D; if two Roots are affirmative and one negative, the greateſt of the Affirmatives cannot be greater than $\sqrt{p + \frac{1}{4}bb} + \frac{1}{2}b$, nor leſs than $\sqrt{\frac{1}{9}bb + \frac{1}{3}p} + \frac{1}{3}b$. But the leſſer Root is diminiſhed from this Limit, as the Quantity q is diminiſhed. The Negative is leſs than $\sqrt{\frac{4}{9}bb + \frac{4}{3}p} - \frac{1}{3}p$, but greater than $\sqrt{p + \frac{1}{4}bb} - \frac{1}{2}b$.

But here it muſt be noted, that the negative Roots are every where marked with the affirmative Sign, becauſe theſe are the affirmative Roots of thoſe four Equations, in which there is $+ b$, and q is marked with a contrary Sign, as I obſerved before. The Demonſtration of all theſe follows from hence, that whenever the Center of the Circle R falls upon the Curve-Lines V P X or V Δ L, its Circumference touches the Parabola in a Point whoſe Diſtance from the Axis is $\sqrt{\frac{2}{3}}$ V H. But when the Center falls upon the Line D P D, one of the Roots becomes $= 0$, and therefore the Cubic is reduced to a Quadratic, or to $z^2 - bz + p = 0$, whoſe Roots mark out the Limits where the Quantity q vaniſhes. And the leſs q is, ſo much the nearer the Roots approach to theſe Limits. It is alſo a Quadratic when the Center falls in the Axis; that is, when $\frac{1}{2}q = \frac{1}{6}bp - \frac{1}{27}b^3$, in the firſt Form, or $\frac{1}{2}q = \frac{1}{27}b^3 + \frac{1}{6}bp$ in the ſecond. In the third it is impoſſible, but in the fourth when $\frac{1}{2}q = \frac{1}{27}b^3 + \frac{1}{6}bp$. In which Caſe the leſſer of the affirmative Roots is $\frac{1}{3}b$, the greater $\sqrt{\frac{1}{3}b^2 + p} + \frac{1}{3}b$. And the negative Root is $\sqrt{\frac{1}{3}bb + p} - \frac{1}{3}b$.

In

In the first Formula the Roots are $\frac{1}{3}b$ and $\frac{1}{3}b \pm \sqrt{\frac{1}{3}bb - p}$; in the second $\frac{1}{3}b$ and $\sqrt{\frac{1}{3}bb - p} + \frac{1}{3}b$ are affirmative; but $\sqrt{\frac{1}{2}bb - p}$ $- \frac{1}{3}b$ is negative.

And this seems to be sufficient as to Cubics. But because of the excellent Use of that Method, which finds the Roots of these Equations by means of the Table of Sines; I have thought proper to add an Example or two, to shew the Compendiousness of this Praxis. Let there be proposed this Equation $z^3 - 39z^2 + 479z - 1881 = 0$, of which the Roots z are required. We shall have $\sqrt{\frac{1}{9}bb - \frac{1}{3}p} = \sqrt{9\frac{1}{3}} = \sqrt{d}$, whose double $\sqrt{37\frac{1}{3}}$ is the Radius of the Circle; and $\frac{\frac{1}{27}b^3 + \frac{1}{2}q - \frac{1}{6}bp}{\sqrt{d^3}} = \frac{2197 + 940\frac{1}{2} - 3113\frac{1}{2}}{9\frac{1}{3}\sqrt{9\frac{1}{3}}} = \frac{24}{9\frac{1}{3}\sqrt{9\frac{1}{3}}}$ is the Tabular Sine of the Angle, that is, making the Division by Help of the Logarithms, Log. 9. 9251560, to which answers the Angle $57^\circ\ 19'\ 11''\ \frac{1}{2}$. A third Part of this is $19^\circ\ 6'\ 24''$, and of the Complement $40^\circ\ 53'\ 36''$. The Sines give Log. 9. 514933, and 9. 816011, which drawn into the Radius $\sqrt{37\frac{1}{3}}$, produce Y & and Y &, Log. 0. 301030, $= 2$, and Log. 0. 601059, $= 4$. But the third Y & is equal to their Sum, or 6. Therefore the Roots are $13 - 4 = 9$, $13 - 2 = 11$, and $13 + 6 = 19$; of all of which the foresaid Equation is composed. Where it may be observed, that the two lesser Roots do not exceed $\frac{1}{6}b$, or 13, because in the Construction the Center R falls to the Right Hand of the Axis; that is, $\frac{1}{6}bp$ is less than $\frac{1}{27}b^3 + \frac{1}{2}q$.

For another Example suppose $x^3 - 15x^2 - 229x - 525 = 0$, and let the Roots be sought. We have $\sqrt{\frac{1}{9}bb + \frac{1}{3}p} = \sqrt{101\frac{1}{3}} = \sqrt{d}$, and the Radius of the Circle is $\sqrt{405\frac{1}{3}}$. Then $\frac{\frac{1}{27}b^3 + \frac{1}{6}bp + \frac{1}{2}q}{\sqrt{d^3}} = \frac{125 + 572\frac{1}{2} + 262\frac{1}{2}}{101\frac{1}{3}\sqrt{101\frac{1}{3}}} = \frac{960}{101\frac{1}{3}\sqrt{101\frac{1}{3}}}$ = Tabular Sine of an Arch, whose Log. 9. 9736426, and the Arch itself will be $70^\circ\ 14'\ 22''$, whose third Part is $23^\circ\ 24'\ 47''\ \frac{1}{2}$, and of its Complement $36^\circ\ 35'\ 12''\ \frac{1}{2}$; whose Logarithmic Sines are 9. 599183, and 9. 775275; to which the Log. of $\sqrt{405\frac{1}{3}}$ being added, they become Log. 0. 903089. $= 8$, and Log. 1. 079 81, $=$ 12, and their Sum $= 20$. Hence it is concluded, that $20 \pm \frac{1}{3}b$, or 25,

is

is equal to the affirmative Root, and $8 - \frac{1}{3}b$, or 3, and $12 - \frac{1}{3}b$, or 7, are equal to the negative Roots. Now if the Equation had been $x^3 + 15x^2 - 229x + 525 = 0$, the Roots 3 and 7 would have been affirmative, and 25 negative. But other Cubic Equations, which are explicable by one Root only, are to be reſolved by *Cardan's* Rules after the ſecond Term is taken away; nor can I perceive how it may be done by leſs Calculation. But if this Root were deſired, expreſſed by the Quantities b, p, q; I ſay in the firſt Form it would be $\frac{1}{3}b$, adding or ſubtracting the Sum or Difference of the Cubic Roots of $\sqrt{\frac{1}{4}qq - \frac{1}{108}p^2b^2 + \frac{1}{27}b^3q - \frac{1}{6}bpq + \frac{1}{27}p^3} \pm \frac{1}{27}b^3 + \frac{1}{2}q - \frac{1}{6}bp$; that is, it will be $+$ if $\frac{1}{27}b^3 + \frac{1}{2}q$ is greater than $\frac{1}{6}bp$, otherwiſe $-$. Alſo it will be the Sum as often as $\frac{1}{3}bb$ is greater than p, but if $\frac{1}{3}bb$ is leſs, then the Difference. And in the other Forms the Root is always made up of the ſame Elements, with Variations of the Signs $+$ and $-$, as any one will eaſily perceive who has a Mind to try.

By Help of the Logarithmic Table of Verſed Sines theſe Roots may be found readily enough; that is, if the Numeral Coefficients are Surds or Fractions, and the Roots are effable in Numbers, as it generally happens. Now this is the Rule. In the firſt and ſecond Formula, if $\frac{1}{3}bb$ is leſs than p, make $\frac{1}{3}p - \frac{1}{9}bb = d$, and taking for Radius the Difference between $\frac{1}{6}bp$ and $\frac{1}{27}b^3 + \frac{1}{2}q$, that is HR in the firſt, and between $\frac{1}{6}bp + \frac{1}{2}q$, and $\frac{1}{27}b^3$ in the ſecond; let the Angle be found whoſe Tangent is $d\sqrt{d}$. Then as the Coſine of this Angle to its Verſed Sine, ſo is the Difference taken for Radius to a fourth; whoſe Cubic Root by Triſection will be had for a Logarithm. Then dividing $\frac{1}{3}p - \frac{1}{9}bb$ by this Cubic Root, from the Quotient ſubtract the Diviſor, and the Remainder will be the Quantity Y &c. The Sum of $\frac{1}{3}b$ and this Remainder, if the Center fall on the Right Side the Axis, otherwiſe the Difference of the ſame, will be the Root ſought. Now if $\frac{1}{3}bb$ is greater than p, taking HR for Radius, let $d\sqrt{d}$ or the Diſtance of the Paraboloid from the Axis, be the Sine of of a certain Arch. Let the verſed Sine of this be multiplied by the Radius, or $\frac{1}{6}bp - \frac{1}{27}b^3 \pm \frac{1}{2}q$, and the Logarithm of the Product being triſected, its Cubic Root will be had, by which let $\frac{1}{9}bb - \frac{1}{3}p$ be divided. I ſay that the Sum of the Quotient and Diviſor in the ſame Manner added to or taken from $\frac{1}{3}b$, will exhibit the Root required. And in like manner in the third and fourth Forms, unleſs that $\frac{1}{27}b^3 + \frac{1}{6}bp \pm \frac{1}{2}q$ is to be aſſumed for the Radius, and $\frac{1}{9}bb + \frac{1}{3}p$ into $\sqrt{\frac{1}{9}bb + \frac{1}{3}b}$, or $d\sqrt{d}$ for the Sine. But theſe Precepts will be better underſtood by Examples.

Let the Cubic Equation be $z^3 - 17z^2 + 54z - 350 = 0$, and let the Root z be required. Here $\frac{1}{3}bb$ is greater than p, but q is greater than the Cube of $\frac{1}{3}b$; therefore the Equation is explicable by only one affirmative Root. Now $\frac{289}{9} - \frac{54}{3}$ is d, and $\frac{127}{9}\sqrt{\frac{127}{9}}$ is to be taken for the Sine to Radius $\frac{4913}{27} + 175 - 153$, that is $\frac{5507}{27}$. The Arch belonging to it is $15^\circ : 3' : 49''$. The Logarithm of the verſed Sine of this is 8. 5362376, which added to the Logarithm of the Radius 2. 3095913, gives 0. 8457889, whoſe third Part 0. 2819276 is the Logarithm of the Cubic Root 1. 91394. This dividing $\frac{127}{9}$ or d, the Quotient is 7. 37281. Then the Sum of the Quotient and Diviſor adding $\frac{1}{3}b$ is the Root ſought, or 14, 9534, &c.

Having now done with Cubic Equations, we are to undertake Biquadratics. Theſe always have either none, or two, or four true Roots, the Determination of which depends partly on the Coefficients, and partly on the Sign and Magnitude of the given abſolute Number. In the Conſtruction of the Equation $z^4 - bz^3 + pz^2 - qz + r = 0$, make $BD = \frac{1}{4}b$, $AB = \frac{1}{16}bb$, $BK = \frac{1}{2}$, or half the *latus rectum*, $KC = 2AB = \frac{1}{8}bb$, $KE = \frac{1}{8}bb - \frac{1}{2}p$, $AE - \frac{1}{2} = \frac{3}{16}bb - \frac{1}{2}p$, $FE = \frac{1}{16}b^3 - \frac{1}{4}bp$, and $EG = \frac{1}{16}b^3 - \frac{1}{4}bp + \frac{1}{2}q$. Which being conſtructed a Circle

Fig. 49.

Circle with Center G and Radius $\sqrt{GDq - r}$ will interſect the Parabola, either in none, two, or four Points, which exhibit all the Values of z, or all the Roots, by Perpendiculars upon the Line H D. Now that there may be four, it is plain the Center of the Circle muſt be placed ſomewhere within the Space, from any Point of which three Perpendiculars can be let fall upon the Curve of the Parabola; and at the ſame Time the Radius muſt be leſs than the greateſt of thoſe Perpendiculars, and greater than the middlemoſt. Now if the Center be placed without this Space, ſo that only one Perpendicular can be let fall upon the Parabola, than which the Radius is greater; or if it be leſs than the middlemoſt of the three Perpendiculars, but greater than the leaſt of them, there can be only two Roots. But there will be none as often as the Radius $\sqrt{GDq - r}$ is leſs than the leaſt of the three, or than that one when there is but one. Now what Sort of Space this is, by what Limits it is circumſcribed, and upon what Conditions the Radius of the Circle is leſs or greater than the ſaid Perpendiculars, we muſt now inquire. And firſt it muſt be ſhewn, how a Perpendicular may be let fall upon the Parabola.

Let A B C be a Parabola, A E its Axis, A V its half *latus rectum*, and G the Point from whence the Perpendicular is to be drawn. Make G E perpendicular to the Axis, and let V E be biſected in F. And erecting the Perpendicular F H on the ſame Side of the Axis, make F H $= \frac{1}{4}$ G E. I ſay, that a Circle deſcribed with Center H, and with Radius H A, will interſect the Parabola in three Points, or in one Z, to which the Right Lines G Z being drawn, will inſiſt perpendicularly upon the Curve of the Parabola. *Fig.* 53.

Now that there may be three ſuch Interſections, it is required that the Center of the Circle H ſhould be ſo placed, that it may be within the Space included by the Paraboloids. That is, that F H may be leſs than $\sqrt{\frac{8}{27}}$ V F c, or F H q leſs than the Cube of $\frac{2}{3}$ V F, and therefore G E $=$ 4 F H will be leſs than $\sqrt{\frac{8}{27}}$ V F c or $4\sqrt{\frac{1}{27}}$ V E c, that is, the Square of G E leſs than $\frac{16}{27}$ V E c. Therefore theſe Limits coincide with the two Paraboloids of the ſame Kind with thoſe which we made uſe of in Cubics, but whoſe *latus rectum* is as little again; that is, $\frac{27}{16}$ of the *latus rectum* of the Parabola, or $\frac{27}{8}$ of A V. Therefore it is the very Curve by the Evolution of which the Parabola is generated, as Mr. *Huygens* has demonſtrated, and which the Line D F always touches, which is perpendicular to the Parabola in the Point D. But the Point P, in which the Right Line D F touches the Pa- *Fig.* 52.

raboloid, is the Center of the Circle, which being deſcribed with Radius D P coincides with the Parabola in the Point D, or is of the ſame Degree of Curvity; which is manifeſt of itſelf.

Therefore ſuch Paraboloids V X P, V N Δ, being deſcribed on each Side the Axis, it is plain unleſs the Center of the Circle is fixed within theſe Limits, it cannot cut the Parabola in more than two Points. Whence we may determine under what Conditions the Coefficients of the intermediate Terms are reſtrained in Biquadratic Equations, in order to have four Roots. And firſt it appears that p cannot be greater than $\frac{3}{8}bb$, in thoſe Forms in which it is $+p$, nor q greater than $\frac{1}{16}b^3$. But in general $\frac{1}{16}b^3 \mp \frac{1}{4}pb \mp \frac{1}{2}q$, that is, the Diſtance of the Center from the Axis E G, muſt be leſs than $EH = 4\sqrt{\frac{1}{27}}VEc$; that is, becauſe of $VE = \frac{3}{16}bb \mp \frac{1}{2}p$, leſs than $\frac{1}{4}bb \mp \frac{2}{3}p\sqrt{\frac{1}{16}bb \mp \frac{1}{6}p}$; the Signs $+$ and $-$ being left doubtful, that they may be varied according to the Exigence of any Equation; as we have ſhewn before in Cubics.

But the Limitation of the laſt Term r cannot ſo eaſily be found. And this becauſe to let fall a Perpendicular upon the Curve of a Parabola is a ſolid Problem, and which cannot be reſolved without the Solution of a Cubic Equation. Therefore firſt let the ſecond Term of the Equation be wanting, or if it be preſent let it be taken away, that the Equation may have this Form $z^4 : * : pz^2 : qz : r : = 0$. Now if it be $-r$, the Equation may always be explained by two or four Roots: But that there ſhould be four, the Center of the Circle muſt be had below the foreſaid Paraboloids, or that there may be $-p$, and qq leſs than $\frac{8}{27}p^3$, or than the Cube of $\frac{2}{3}p$. Then let there be found the Roots of this Equation, $y^3 : * : \frac{1}{2}py. \frac{1}{4}q = 0$, the ſame Signs being annexed to p and q as in the Biquadratic Equation. Now theſe Roots may eaſily be found by help of the Table of Sines. Thoſe three Values of y being found, (which are the Ordinates to the Axis of the Parabola, from the Points where the Perpendiculars fall upon the Curve, that is Z Y,) the Quantity $py^2 - 3y^4$ compoſed of the leſſer Root y will expreſs the greateſt Value of r, if it is $-r$; than which if r is leſs, the Equation will have four Roots, otherwiſe but two. But if it is $+r$, it muſt be leſs than $3y^4 - py^2$, compoſed of the middlemoſt y; for if it be greater, there can be but two Roots, at leaſt if r be leſs than $3y^4 - py^2$ compoſed of the greateſt y. But if it be greater than this, the Equation is explicable by no true Root at all. Now theſe ſame Limits may be otherwiſe

marked

marked out by the Quantity q. That is, $\frac{1}{2}qy - y^4$ in the firſt Caſe, $y^4 - \frac{1}{2}qy$ in the ſecond, and $y^4 + \frac{1}{2}qy$ in the third.

But it may happen that the two leſſer Quantities y are not far aſunder, ſo that each of the Perpendiculars may be greater than the Right Line G A, that is, when qq is greater than $\frac{4}{27}p^3$, but leſs than $\frac{8}{27}p^3$, the Center falling within the Space intercepted by the Paraboloids, Fig. 51, 52. In this Caſe if it is $+r$, there can be but two Roots, the Quantity $y^4 + \frac{1}{2}qy$ compoſed of the greateſt y being greater than r, otherwiſe none at all. But if $\frac{1}{2}qy - y^4$ compoſed of the leaſt y is greater than r having the Sign $-$, and r is greater than $\frac{1}{2}qy - y^4$ compoſed of the mean y, then there would be four Roots; and but two only if r ſhould be either greater than the former or leſs than the latter.

But in the Equation if it ſhould be $+p$, or if it is $-p$ and qq greater than $\frac{8}{27}p^3$. the Equation $y^3 : * : \frac{1}{2}py : \frac{1}{4}q = 0$, is explicable only by one Root y, that is, one Perpendicular alone can be let fall from the Center of the Circle. Whence we may certainly conclude, that there can be had only two Roots from the given Equation, the Sum of which, if it is $-r$, increaſes with the Quantity r; but if it is $+r$, the Quantity y being found, that Quantity r muſt be leſs than $y^4 + \frac{1}{2}qr$. For if it be greater, the propoſed Equation will be abſurd and impoſſible.

It would be a tedious and unneceſſary Thing to run through all the Equations of this Sort, ſince it muſt be evident to any one that conſiders it, what Roots are negative and what affirmative; and that the Limits of theſe Roots may be had from the Quantities y being found. Yet for an Example, which may be imitated in others, let it be propoſed to find the Limits or Conditions, under which in a Biquadratic Equation four affirmative Roots may be had. Now this will happen as often as the Center of the Circle G is placed in the Space V P K, and at the ſame Time there is $+r$, or the Radius of the Circle is leſs than G D. Whence it is plain that the Equation we are treating of muſt be of this Form $z^4 - bz^3 + pz^2 - qz + r = 0$. But p cannot be greater than $\frac{3}{8}bb$, nor in this Caſe $\frac{1}{4}pb$ greater than $\frac{1}{16}b^3 + \frac{1}{2}q$. Alſo it is neceſſary that $\frac{1}{4}bb - \frac{2}{3}p$ drawn into $\sqrt{\frac{1}{16}bb - \frac{1}{6}p}$ muſt be greater than $\frac{1}{6}b^3 + \frac{1}{2}q - \frac{1}{4}pb$; and from theſe Limitations Fig. 52.

it

it will be certain, that the Center will be found within the Space VPK. Now that the Quantity r may be determined, we must first solve this Cubic Equation $y^3 : * : -\frac{3}{16}b^2 - \frac{1}{2}py = \frac{1}{32}b^3 + \frac{1}{4}q - \frac{1}{8}pb$, and the Points will be had, upon which the Perpendiculars from the Center will fall upon the Curve of the Parabola.

Now the three Values of this y being found, r must be less than $\frac{3}{256}b^4 + \frac{1}{4}bq - \frac{1}{16}b^2p + 3y^4 - \frac{3}{8}b^2y^2 + py^2$, composed of the middle Value of y, and greater than $\frac{3}{256}b^4 + \frac{1}{4}bq - \frac{1}{16}b^2p + 3y^4 - \frac{3}{8}b^2y^2 + py^2$, when composed of the least y. But if r exceeds these Limits, there can be only two Roots. Lastly if $\frac{3}{256}b^4 + \frac{1}{4}bq - \frac{4}{16}b^2p + 3y^4 - \frac{3}{8}b^2y^2 + py^2$, when composed of the greatest y, shall be less than r, the Equation is impossible.

It may also happen that there are four affirmative Roots, when the Center G is placed in the little Space V T S, drawing R T S perpendicular upon the middle of the Line A D. This happens when p is greater than $\frac{5}{16}bb$, and $\frac{1}{4}bb - \frac{2}{3}p\sqrt{\frac{1}{16}bb - \frac{1}{6}p}$ is greater than $\frac{1}{8}pb - \frac{5}{128}b^3 - \frac{1}{2}q$. In which Case always two, and sometimes three, of the Roots are greater than $\frac{1}{4}b$.

Fig. 52. But here it must be observed, that Limit produced from the least y is sometimes negative, or less than nothing; that is, whenever the greatest of the three Perpendiculars is greater than G D. This may be diminished to nothing, if it happens to be $+r$ from the prescribed Limit composed of the middlemost y. And the Defect of the Limit from the least y shews, how great $-r$ may be in the Equation, if there are three affirmative Roots in the Equation and one Negative: Which if it exceeds, there are only two Roots, one affirmative and one negative. All these Things are demonstrated from hence, that the aforesaid Limits of the Quantity r are the Differences of the Squares of the Line G D and of the Perpendiculars upon the Curve of the Parabola.

But because of the intricate Restrictions which the Diversity of Signs produces in these Equations, it is always best to take away the second Term, and then

then by the Precepts already delivered to find the Number of the Roots and their Signs; especially if those Quantities y are not far remote from one another. Now of these four affirmative Roots, two are always less than $\frac{1}{4}b$, and two greater; that is, if D G is less than A G, or $\frac{1}{4}pb$ than $\frac{3}{64}b^3+q$. But three of them are less than $\frac{1}{4}b$, as often as the middlemost Perpendicular, or that found from the middlemost y, is greater than A G, or $\frac{3}{8}bb$ greater than $3y^3-py^2$ computed from the same middlemost y. But the fourth and greatest Root is greater than the greatest $y+\frac{1}{4}b$, and is equal to the Difference of b and the Sum of the other three Roots, and therefore is less than b. But I shall proceed no farther. Perhaps those who have a greater Insight into the Nature of the Parabola may be able to draw all these Conclusions more compendiously; but it may well be doubted, whether all these Quantities, b, p, q, and r, can be rightly determined, without the Solution of a Cubic Equation. For whatever can be done in this Matter by plane Equations, can only exhibit Approximations, and not the true Limits.

XX. Mr. *de Lagney* has lately given us two very compendious Rules for an Approximation to the Cubic Root; one of them rational, and the other irrational. That is, the Root of the Cube a^3+b is between $a+\frac{ab}{3a^3+b}$, and $\sqrt{\frac{1}{4}a^2+\frac{b}{3a}}+\frac{1}{2}a$. And the Root of the fifth Power a^5+b he thus expresses, $\frac{1}{2}a+\sqrt{\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}-\frac{1}{4}a^2}$. Now the aforesaid Rules are demonstrated from the Genesis of the Cube and of the fifth Power. For supposing the Side of any Cube to be $a+e$, the Cube derived from thence will be $a^3+3a^2e+3ae^2+e^3$. So that if a^3 be supposed a Cubic Number which is the next less to a non-Cubic, e^3 will be less than Unity, and the Remainder or b will be equal to the other Members of the Cube $3a^2e+3ae^2+e^3$. And e^3 being rejected because of its Smallness, it will be $b=3a^2e+3ae^2$. And since a^2e is much greater than ae^2, $\frac{b}{3a^2}$ will not much exceed e; and putting $e=\frac{b}{3a^2}$, $\frac{b}{3a^2+3ae}$ (to which the Quantity e is nearly equal) will be found equal to $\frac{b}{3a^2+\frac{3ab}{3a^2}}=\frac{b}{3a^2+\frac{b}{a}}=\frac{ab}{3a^3+b}=e$ *ferè*. So that the Side of the Cube a^3+b is equal to $a+e=a+\frac{ab}{}$

The Extraction of all Roots without any previous Reduction; by Mr. Ed. Halley. N. 210. p. 136. May, An. 1694.

$\frac{a b}{3 a^3 + b}$, nearly, which is the Rational Form of Mr. *de Lagney*. Now if a^3 be a Cubic Number next greater than the given Number, the Side of the Cube $a^3 - b$ by a like Way of reaſoning will be found $a - \frac{a b}{3 a^3 - b}$. And this Approximation to the Cubic Root, which is very eaſy and expeditious, makes but a very ſmall Error in defect, ſince e the Remainder of the Root found by this Means, will be but little leſs than the true Remainder. Now the irrational Form may likewiſe be eaſily derived after the ſame Manner. For becauſe $b = 3 a^2 e + 3 a e^2$, or $\frac{b}{3 a} = a e + e e$, and therefore $\sqrt{\frac{1}{4} a^2 - \frac{b}{3 a}} = \frac{1}{2} a + e$, and $\sqrt{\frac{1}{4} a^2 - \frac{b}{3 a}} + \frac{1}{2} a = a + e$ which is the Root required. And after the ſame Manner will be had $\frac{1}{2} a + \sqrt{\frac{1}{4} a^2 - \frac{b}{3 a}}$ for the Side of the Cube $a^3 - b$. Now this Form approaches ſomething nearer to the true Root than the rational Form, erring in Exceſs as the other does in Defect, and ſeems to be more convenient for Practice, ſince the Reſtitution of the Calculation is nothing elſe but the continual Addition or Subtraction of the Quantity $\frac{e^3}{3 a}$, as the little Supplement e becomes known. So that it may rather be wrote $\sqrt{\frac{1}{4} a^2 + \frac{b - e^3}{3 a}} + \frac{1}{2} a$ in the former Caſe, and $\frac{1}{2} a + \sqrt{\frac{1}{4} a^2 + \frac{e^3 - b}{3 a}}$ in the latter. Now by either Form the Figures already known in extracting the Root are at leaſt tripled, which I believe will be a very acceptable Compendium to Arithmeticians, and therefore I congratulate the Inventor upon it. But that the Uſefulneſs of theſe Rules may be the better perceived, I ſhall add an Example or two.

Example I. Let the Side of the double Cube be ſought, or make $a^3 + b = 2$. Hence $a = 1$, $b = 1$, and $\frac{b}{3 a} = \frac{1}{3}$, and therefore $\frac{1}{2} + \sqrt{\frac{7}{12}}$, or 1, 26 will be found near the true Root. But the Cube of 1, 26 is 2, 000376, and therefore $0,63 + \sqrt{0,3969 - \frac{0,000376}{3,78}}$ or $0,63 + \sqrt{0,3968005291005291} = 1,259921049895 -$; which exhibits the Side of the double Cube as far as thirteen Figures with little Trouble, that is, by one Diviſion and Extraction of the Square-root; whereas had it been found by the common Way of Operation, every Arithmetician knows what Labour it muſt have coſt him. Now this Calculation may be continued at Pleaſure, by by increasing the Square by the Addition of $\frac{e^3}{3a}$: Which Correction in this Case brings only the Increase of an Unit to the fourteenth Figure.

Example II. Let the Side of a Cube be sought, which is equal in Measure to a Gallon, containing 231 solid Inches. The next lesser Cube is 216, whose Cubic Root is $6 = a$, and the Remainder $15 = b$. Therefore for the first Approximation we have $3 + \sqrt{9 + \frac{5}{6}} =$ to the Root. And because $\sqrt{9,8333\ldots} = 3,1358\ldots$ it is plain that $6,1358 = a + e$. Now make $6,1358 = a$, and we shall have its Cube 231,000853894712, and according to the Rule $3,0679 + \sqrt{9,41201041 - \frac{0,000853894712}{18,4074}}$ $= 6,13579243966195897$, which is the Side of the given Cube very exactly, being true in the eighteenth Figure, and falling short in the nineteenth; which Calculation I performed within an Hour's Time. Now this Form is deservedly to be preferred before the rational one, which, because of its large Divisor, cannot be managed without a great deal of Trouble; whereas the Extraction of the Square-root proceeds much more easily, as manifold Experience has informed me.

Now the Rule for the Root of the pure Sursolid, or of the fifth Power, is of a little higher Inquiry, and yet performs the Matter much more perfectly. For it quintuples at least the given Figures of the Root, nor does it require much or very operose Calculation. But the Author has no where given his Method of Investigation or Demonstration of it, tho' it seemed most to be wanting: Especially as it is faultily printed in his Book, which may easily mislead the Unskilful. Now the fifth Power of the Side $a + e$ is made up of these Members following, $a^5 + 5a^4e + 10a^3e^2 + 10a^2e^3 + 5ae^4 + e^5 = a^5 + b$; whence $b = 5a^4e + 10a^3e^2 + 10a^2e^3 + 5ae^4$; (rejecting e^5 because of its Smallness) wherefore $\frac{b}{5a} = a^3e + 2a^2e^2 + 2ae^3 + e^4$, and adding $\frac{1}{4}a^4$ on each Side, and extracting the Square-root, it will be $\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} = (\sqrt{\frac{1}{4}a^4 + a^3e + 2a^2e^2 + 2ae^3 + e^4} =) \frac{1}{2}a^2 + ae + e^2$. Then subtracting $\frac{1}{4}a^2$ from each Side, and extracting the Square-root again, $\sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}a^2} = (\sqrt{\frac{1}{4}a^2 + ae + e^2} =)$ $\frac{1}{2}a + e$, to which if you add $\frac{1}{2}a$, 'tis $a + e = \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}a^2}$ $=$ Root of the Power $a^5 + b$. Now if it had been $a^5 - b$, (assuming a greater

greater than it ought to be) the Rule would have been thus, $\frac{1}{2}a +$ $\sqrt{\sqrt{\frac{1}{4}a^4 - \frac{b}{5a}} - \frac{1}{4}aa} =$ Root of $a^5 - b$.

And this Rule approximates wonderfully, ſo that there can ſcarce be any Need of Reſtitution. And upon farther conſidering this Matter, I have fallen upon a certain general Method of Forms for any Power, which is elegant enough, and which I cannot prevail upon myſelf to conceal; ſince in the higher Powers they triplicate at leaſt the known Figures of the Root.

Now theſe Forms proceed after this Manner, as well the rational as the irrational.

$$\sqrt[2]{a^2 + b} = \tfrac{0}{1}a + \sqrt{\tfrac{1}{1}aa + \frac{b}{1a^0}}, \text{ or } a + \frac{ab}{2a^2 + \frac{1}{2}b},$$
$$\sqrt[3]{a^3 + b} = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + \frac{b}{3a}}, \text{ or } a + \frac{ab}{3a^3 + \frac{2}{2}b},$$
$$\sqrt[4]{a^4 + b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa + \frac{b}{6a^2}}, \text{ or } a + \frac{ab}{4a^4 + \frac{3}{2}b},$$
$$\sqrt[5]{a^5 + b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa + \frac{b}{10a^3}}, \text{ or } a + \frac{ab}{5a^5 + \frac{4}{2}b},$$
$$\sqrt[6]{a^6 + b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa + \frac{b}{15a^4}}, \text{ or } a + \frac{ab}{6a^6 + \frac{5}{2}b}, \text{ \&c.}$$

And ſo in other higher Powers. Now if a ſhould be aſſumed greater than the Root ſought, (which would be done with Advantage as often as the Power to be reſolved is much nearer to the Power of the integer Number that is next greater, than it is to the next leſſer) changing what is to be changed, the ſame Expreſſions of the Roots ariſe.

$$\sqrt[2]{a^2 - b} = \tfrac{0}{1}a + \sqrt{\tfrac{1}{1}aa - \frac{b}{1a^0}}, \text{ or } - \frac{ab}{2a^2 - \frac{1}{2}b},$$
$$\sqrt[3]{a^2 - b} = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - \frac{b}{3a}}, \text{ or } - \frac{ab}{3a^3 - \frac{2}{2}b},$$
$$\sqrt[4]{a^2 - b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa - \frac{b}{6a^2}}, \text{ or } - \frac{ab}{4a^4 - \frac{3}{2}b},$$
$$\sqrt[5]{a^2 - b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa - \frac{b}{10a^3}}, \text{ or } - \frac{ab}{5a^5 - \frac{4}{2}b},$$
$$\sqrt[6]{a^2 - b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa - \frac{b}{15a^4}}, \text{ or } - \frac{ab}{6a^6 - \frac{5}{2}b}, \text{ \&c.}$$

Now

Now between theſe two Limits the true Root always conſiſts, being ſomething nearer the irrational than the rational. But e found according to the irrational Formula always errs in Exceſs, as the Quote reſulting from the rational always errs in Defect; therefore if it is $+b$, the irrational gives the Root greater than the Truth, and the rational leſs. And juſt the contrary if it be $-b$. And this may ſuffice concerning the finding of the Roots of pure Powers, which for ordinary Uſes (and accurately enough) may be done more eaſily by Help of the Logarithms. But whenever the Root is to be extracted very accurately, and beyond the Reach of the Logarithms, we muſt neceſſarily have Recourſe to ſuch Methods as theſe. Beſides, as from the Diſcovery and Conſideration of theſe Forms, I have hit upon an univerſal Rule for affected Equations, which I believe will be made uſe of with good Advantage by all ſuch as are ſtudious of Geometry and Algebra; I was willing to lay open with all poſſible Perſpicuity the Foundation of this Diſcovery.

About the Year 1687 I made publick a Method then newly invented by me, which contained a neat and very eaſy general Conſtruction of affected Equations, not beyond Quadrato-quadratics; ſince which Time I have always had a very ſtrong Deſire of performing the ſame in Numbers. A little after that Mr. *Raphſon* ſeemed in a good Meaſure to have accompliſhed my Wiſhes; till Mr. *de Lagney* ſhewed me, by his little Book on this Subject, that the Thing might be ſtill performed more compendiouſly. Now my Method is this.

Let the Root of any Equation z be conceived to be compoſed of two Parts, $a +$ or $- e$, of which let a be aſſumed (by Suppoſition) as near as may be to the Quantity z, (which is convenient indeed, but not abſolutely neceſſary) and of the Quantity $a \pm e$ let there be formed all the Powers, to which z ariſes in the given Equation, to which let their Numeral Coefficients be affixed reſpectively. Then let the Power to be reſolved be ſubtracted from the Sum of the given Parts in the firſt Column, where e is not found, which is called the *Homogeneum Comparationis*, and let the Difference be $\pm b$. Then let there be found the Sum of all the Coefficients of the Side e in the ſecond Column, which let be s. Laſtly, in the third Column, let all the Coefficients be added of the Square ee, the Sum of which we may call t. Then the Root ſought z will be had in the rational Form $z = a \pm \frac{sb}{ss \pm tb}$, and in the irrational Form $z = a \pm \frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}ss \pm bt}}{t}$; which it may be worth while to illuſtrate by Examples. But as a convenient Help it may be proper to have at Hand a general Table, exhibiting the Geneſis of all the Powers of $a \pm e$, which may be eaſily continued farther if neceſſary. I will carry it on to the ſeventh Power, becauſe few Problems go beyond that. This Table may juſtly be called a general Analytical Speculum. Now the aforeſaid Powers, ariſing from the continual Multiplication of $z = a + e$, are as follows with their Coefficients adjoined.

$$\begin{array}{l}
cz = ca + 1\,c a^0 e \\
dz^2 = da^2 + 2\,d a^1 e + 1\,d a^0 e^2 \\
fz^3 = fa^3 + 3\,f a^2 e + 3\,f a^1 e^2 + 1\,f a^0 e^3 \\
gz^4 = ga^4 + 4\,g a^3 e + 6\,g a^2 e^2 + 4\,g a^1 e^3 + 1\,g a^0 e^4 \\
hz^5 = ha^5 + 5\,h a^4 e + 10\,h a^3 e^2 + 10\,h a^2 e^3 + 5\,h a^1 e^4 + 1\,h a^0 e^5 \\
kz^6 = ka^6 + 6\,k a^5 e + 15\,k a^4 e^2 + 20\,k a^3 e^3 + 15\,k a^2 e^4 + 6\,k a^1 e^5 + 1\,k a^0 e^5 \\
lz^7 = la^7 + 7\,l a^6 e + 21\,l a^5 e^2 + 35\,l a^4 e^3 + 35\,l a^3 e^4 + 21\,l a^2 e^5 + 7\,l a^1 e^6 + 1\,l a^0 e^7 \\
\qquad\quad s \qquad\quad t \qquad\quad u \qquad\quad w \qquad\quad x \qquad\quad y
\end{array}$$

Now if it ſhould be $a - e = z$, the Table would be compoſed of the ſame Parts, but only the odd Powers of e muſt be negative, as e, e^3, e^5, *&c.* and the even Powers e^2, e^4, e^6, *&c.* muſt be ſtill affirmative. Let the Sum of the Coefficients of the Side e be denoted by s, of the Square e^2 by t, of the Cube e^3 by u, of the Biquadrate e^4 by w, of the Surſolid e^5 by x, of the Cubo-cube e^6 be y; and ſo on. And as e is ſuppoſed to be but a ſmall Part of the Root required, all the Powers of e become much leſs than the like Powers of a, and therefore for the firſt Poſition the higher Powers may be rejected, as has been ſhewn in pure Powers; and a new Equation being formed, we ſhall have, as ſaid before, $\pm b = \pm s e \pm t e^2$. For the better underſtanding of which take the following Examples:

Example I. Let the Equation propoſed be $z^4 - 3 z^2 + 75 z = 1000$. For the firſt Suppoſition make $a = 10$, and the following Equation will ariſe.

$$\begin{array}{rcl}
z^4 & = & + a^4 \pm 4 a^3 e + 6 a^2 e^2 \pm 4 a e^3 + e^4 \\
- d z^2 & = & - d a^2 \mp 2 d a e - d e^2 \\
+ c z & = & + c a \pm c e
\end{array}$$

$$\begin{array}{rl}
= & + 10000 \pm 400 e + 600 e^2 \pm 40 e^3 + e^4 \\
& - 300 \mp 60 e - 3 e^2 \\
& + 750 \pm 75 e \\
& - 10000 \\
\hline
& + 450 \pm 415 e + 597 e^2 \pm 40 e^3 + e^4 = 0, \\
& \qquad\qquad s \qquad\quad t \qquad\quad u
\end{array}$$

The Signs $\pm$ are left in Doubt in reſpect of e and e^3, till it is known whether e is negative or affirmative, which may admit of ſome Difficulty, ſince in Equations that have many Roots, the *Homogeneum Comparationis* (as it is called) may be increaſed by diminiſhing the Quantity a, and contrary-wiſe may be diminiſhed when it is increaſed. But the Sign of e is determined by the Sign of the Quantity b. For the Reſolvend being taken away from the *Homogeneum* formed by a, the Sign of $s\,e$, and therefore of the Parts prevailing in its Compoſition, will always be contrary to the Sign of the Difference b. Whence it will appear whether it ſhould be $- e$ or $+ e$, or whether a is greater or leſs than the true aſſumed Root. But e is always equal

equal to $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, whenever b and t have the fame Sign. But when they are connected with different Signs, then the fame e is $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$. Now after it is found that it muft be $-e$, in the affirmative Parts of the Equation let e, e^3, e^5, *&c.* be made negative, and in the negative let them be made affirmative, that is, let them all be wrote with a contrary Sign. But if it be $+e$, let their own Signs continue. Now in our Example we have 10450 inftead of the Refolvend 10000, or $b = + 450$; whence it appears that a was taken above the true Root, and therefore it muft be $-e$. Hence the Equation becomes $10450 - 4015e + 597e^2 - 40e^3 + e^4 = 10000$, that is, $450 - 4015e + 597e^2 = 0$. And therefore $450 = 4015e - 597e^2$, or $b = se - te^2$, whofe Root e is $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, or if you had rather $\frac{s}{2t} - \sqrt{\frac{ss}{4tt} - \frac{b}{t}}$; that is, in the prefent Cafe $e = \frac{2007\frac{1}{2} - \sqrt{3761406\frac{1}{4}}}{597}$, whence arifes the Root fought near the Truth $z = 9,886$. Then I fubftitute this for a fecond Hypothefis, and there arifes $a + e = z = 9,8862603936495\ldots$ very accurately, hardly exceeding the Truth by 2 in the laft Figure; that is, when $\frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t} = e$. And even this, if there were Occafion, might be verified much farther, by fubtracting $\frac{\frac{1}{2}ue^3 + \frac{1}{4}e^4}{\sqrt{\frac{1}{4}ss + tb}}$ if it is $+e$, or adding $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - tb}}$ if it be $-e$, from or to the Root before found: The Compendium of which is fo much the more to be valued, that fometimes from the firft Suppofition alone, but always from the fecond, you may continue the Calculation as far as you pleafe, by keeping to the fame Coefficients. But the foregoing Equation has alfo a negative Root, which is $z = -10,26\ldots$ which any one that pleafes may purfue.

Exam-

Example II. Let $z^3 - 17z^2 + 54z = 350$, and make $a = 10$. Then according to the Rule.

$$\begin{array}{rl} z^3 = & a^3 + 3a^2e + 3ae^2 + e^3 \\ -dz^2 = & da^2 - 2dae - de^2 \\ +cz = & ca + ce \end{array}$$

$$\begin{array}{lllll} & b & s & t & \\ \text{That is,} & +1000 & +300e & +30e^2 & +e^3 \\ & -1700 & -340e & -17e^2 & \\ & +540 & +54e & & \\ & -350 & & & \\ \hline \text{Or,} & -510 & +14e & +13e^2 & +e^3 = 0. \end{array}$$

Now since there is -510, it is plain that a was assumed less than the Truth, and therefore e is affirmative. And from $510 = 14e + 13e^2$, there arises $\frac{\sqrt{bt + \frac{1}{4}ss} - \frac{1}{2}s}{t} = e = \frac{\sqrt{6679} - 7}{13}$; whence 'tis $z = 15,7 \ldots$ which is too much, because a was not taken near enough. Therefore, secondly, let us suppose $a = 15$, and by the same Way of arguing we shall have $e = \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - tb}}{t} = \frac{109\frac{1}{2} - \sqrt{11710\frac{1}{4}}}{28}$; and therefore $z = 14,954068$. Now if we would renew the Calculation a third Time, we should find the Root true to 25 Figures. But if we are contented with fewer, by writing $tb \pm te^3$ instead of tb, or by adding or subtracting $\frac{\frac{1}{2}e^3}{\sqrt{\frac{1}{4}ss \mp tb}}$ to or from the Root before found, we shall soon arrive at our Purpose. Now the proposed Equation cannot be explained by any other Root, because the Power to be resolved 350 is greater than the Cube of $\frac{17}{3}$, or $\frac{1}{3}d$.

Example III. Let it be that Equation which the learned *Wallis*, in Chap. 62. of his Algebra, makes use of in the Resolution of a most difficult Arithmetical Problem, in which he has attained the Root very accurately by the Method of *Vieta*. And the abovementioned Mr. *Raphson*, Pag. 25, 26. brings the same Equation as an Example of his Method. The Equation is $z^4 - 80z^3 + 1998z^2 - 14937z + 5000 = 0$. Now this Equation is of such a Form, as that it may have several affirmative Roots, and what increases the Difficulty, the Coefficients are very great in respect of the given Resolvend. Now that it may be managed the better, let it be divided, and placed according to the known Rules of Punctuation, $-z^4 + 8z^3 - 20z^2 + 15z = 0,5$; where z is but one tenth Part of z in the Equation proposed. Then for the first Supposition let us make $a = 1$. Wherefore $+2 - 5e - 2e^2 + 4e^3 - e^4 - 0,5 = 0$. That is, $1\frac{1}{2} = 5e + 2e^2$,

$2e^2$, and hence $\frac{\sqrt{\frac{1}{4}ss+bt}-\frac{1}{2}s}{t}=e=\frac{\sqrt{37}-5}{4}$, and therefore $z=$ 1,27. Hence it apppears, that 12,7 is the Root of the propoſed Equation pretty near the Truth. Now, in the ſecond Place, let it be ſuppoſed, that $z=12,7$, and according to the Table of Powers,

$$\begin{array}{lllll}
-\ 26014,4641 & -\ 8193,532\,e & -\ 967,74\,e^2 & -\ 50,8\,e^3 & -\ e^4 \\
+\ 163870,640. & +\ 38709,60.\,e & +\ 3048,..\,e^2 & +\ 80,.\,e^3 & \\
-\ 322257,42.. & -\ 50749,2..\,e & -\ 1998,..\,e^2 & & \\
+\ 189699,9... & +\ 14937,...\,e & & & \\
-\ \ \ 5000,.... & & & & \\
\hline
+\ 298,6559 & -\ 5296,132\,e & +\ 82,26\,e^2 & +\ 29,2\,e^3 & -\ e^4=0 \\
\quad b & \quad s & \quad t & \quad u &
\end{array}$$

Therefore $-298,6559=-5296,132\,e+82,26\,e^2$, whoſe Root e according to Rule is $\frac{\frac{1}{2}s-\sqrt{\frac{1}{4}ss-bt}}{t}=\frac{2648,066-\sqrt{6987686,106022}}{82,26}$ $=0,05644080331\ldots=e$ leſs than the Truth. Now that it may be corrected, $\frac{\frac{1}{2}ue^3-e^4}{\sqrt{\frac{1}{4}ss-bt}}$ or $\frac{0,0026201\ldots}{2643,423\ldots}$ becomes 0,00000099117, and therefore e corrected will be 0,05644179448. Now if more Figures of the Root be wanted, of e corrected let there be formed $tue^3-te^4=$ 0,43105602423 . . . . ., then $\frac{\frac{1}{2}s-\sqrt{\frac{1}{4}ss-bt-tue^3+te^4}}{t}$, or $\frac{2648,066-\sqrt{6987685,67496597577\ldots}}{82,26}=0,056441794480744O2$ $=e$. Whence $a+e=z$, or the Root very accurately comes out 12,75644179448074402, ſuch as was found by Dr. *Wallis* in the Place above quoted. Here it may be obſerved, that the renewing of the Calculus always triplicates the true Figures in the aſſumed a, which the firſt Correction, or $\frac{\frac{1}{2}ue^3-\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss-bt}}$, makes quintuple, which Operation is eaſily performed by the Logarithms. The other Correction after the firſt alſo adds a double Number of Figures, ſo that in the whole it makes the aſſumed Figures ſevenfold. But the firſt generally is abundantly ſufficient for all the Uſes of Arithmetick. But what is here ſaid about the Number of Figures rightly aſſumed in the Root, I would have ſo underſtood, that when a is diſtant from the true Root not above a tenth Part, the firſt Figure may be rightly aſſumed; if within an hundredth Part, the two firſt Figures; if within a thouſandth Part, the three firſt Figures may be true. Then when managed according to our Rule, the true Figures will immediately become nine.

It

It remains that I ſhould add a Word or two concerning our rational Form, $e = \frac{sb}{ss \pm tb}$, which will ſeem expeditious enough, and not much inferior to the former, ſince it is able to triplicate the given Figures. For an Equation being formed of $a \mp e = z$, as before, it will ſoon appear whether the aſſumed a be greater or leſs than the Truth, ſince se muſt always have a Sign contrary to the Sign of the Difference of the Reſolvend, and of its *Homogeneum* produced of a. Then ſuppoſing that $\pm b \mp se \pm te = 0$, the Diviſor becomes $ss - tb$ as often as b and t have the ſame Sign. But it becomes $ss + bt$ if their Signs are different. But it ſeems more accommodated to Practice if the Theorem were written $e = \frac{b}{s \pm \frac{tb}{s}}$; for then the Thing would be performed by one Multiplication and two Diviſions, which otherwiſe would require three Multiplications and one Diviſion. Let us alſo take an Example of this Method from the Root of the foregoing Equation 12, 7 . . . in which $\underset{+b}{298,6559}, \underset{-s}{-5296,132\,e} \underset{+t}{+82,26\,e^2} \underset{+u}{+29,2\,e^3} - e^4 = 0$, and therefore $\frac{b}{s - \frac{tb}{s}} = e$. That is, let it be s to t ſo b to $\frac{tb}{s} = 5296$, 132) 298, 6559 into 82, 26 (4, 63875. Wherefore the Diviſor becomes $s = \frac{tb}{s} = 5291, 49325 \ldots$) 298, 6559 (0, 056441 . . . $= e$, that is, five true Figures added to the aſſumed Root. But this Formula cannot be corrected as the foregoing irrational one; ſo that if more Figures of the Root are deſired, it is better to repeat the Calculation by making a new Aſſumption; and the new Quotient, by triplicating the known Figures in the Root, will abundantly ſatisfy even the moſt ſcrupulous Computer.

A Method of Raiſing an Infinite Multinomial to any given Power; by M. de Moivre. N. 230. *p.* 619. Jul. An. 1697.

XXI. *Theorem.*] $\overline{az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + gz^7 + hz^8 + iz^9 \,\&c.}^m = a^m z^m + \frac{m}{1} a^{m-1} b z^{m+1}$

$$+ \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 z^{m+2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 z^{m+3}$$

$$+ \frac{m}{1} a^{m-1} e \qquad\qquad + \frac{m}{1} \times \frac{m-1}{1} a^{m-2} bc$$

$$\times \frac{m}{1} a^{m-1} d$$

$+ \frac{m}{1}$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\,a^{m-4}\,b^{4}\,z^{m+4}$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}\,a^{m-3}\,b^{2}\,c$$

$$+\frac{m}{1}\times\frac{m-1}{1}\,a^{m-2}\,b\,d$$

$$+\frac{m}{1}\times\frac{m-1}{2}\,a^{m-2}\,c^{2}.$$

$$+\frac{m}{1}\,a^{m-1}\,e$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\,a^{m-5}\,b^{5}\,z^{m+1}.$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{1}\,a^{m-4}\,b^{3}\,c$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}\,a^{m-3}\,b^{2}\,d$$

$$+\frac{m}{1}\times\frac{m-1}{1}\times\frac{m-2}{2}\,a^{m-3}\,b\,c^{2}$$

$$+\frac{m}{1}\times\frac{m-2}{1}\,a^{m-2}\,b\,c$$

$$+\frac{m}{1}\times\frac{m-1}{1}\,a^{m-2}\,c\,d$$

$$+\frac{m}{1}\,a^{m-1}\,f$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times\frac{m-5}{6}\,a^{m-6}\,b^{6}\,z^{m+6}$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{1}\,a^{m-5}\,b^{2}\,c$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{1}\,a^{m-4}\,b^{3}\,d$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}\times\frac{m-3}{2}\,a^{m-4}\,b^{2}\,c^{3}$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}\,a^{m\ 3}\,b^{2}\,e$$

$$+\frac{m}{1}\times\frac{m-1}{1}\times\frac{m-2}{1}\,a^{m-3}\,b\,c\,d$$

$$+\frac{m-1}{1}\times\frac{m-1}{1}\,a^{m-2}\,b\,f$$

$$+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\,a^{m-3}\,c^{3}$$

$$+\frac{m}{1}\times\frac{m-1}{1}\,a^{m-2}\,c\,e$$

$$+\frac{m}{1}\times\frac{m-1}{2}\,a^{m-2}\,d^{2}$$

$$+\frac{m}{1}\,a^{m-1}\,g,\ \text{\&c.}$$

I ſuppoſe that the Infinite Number Multinomial is $az + bzz + cz^3 + dz^4 + ez^5$, *&c.* m is the Index of the Power, to which this Multinomial ought to be raiſed; or, if you will, 'tis the Index of the Root which is to be extracted: I ſay, That this Power or Root of the Multinomial is ſuch a Series as I have expreſſed.

For the underſtanding of it, it is only neceſſary to conſider all the Terms by which the ſame Power of z is multiplied; in order thereto, I diſtinguiſh two things in each of theſe Terms; 1°. The Product of certain Powers of the Quantities, a, b, c, d, *&c.* 2°. The Unciæ (as *Oughtred* calls them) prefixed to theſe Products. To find all the Products belonging to the ſame Power of z, to that Product, for Inſtance, whoſe Index is $m + r$ (where r may denote any Integer Number) I divide theſe Products into ſeveral Claſſes; thoſe which immediately after ſome certain Power of a (by which all theſe Products begin) have b, I call Products of the firſt Claſſis; for Example, $a^{m-4}\,b^3\,e$ is the Product of the firſt Claſſis, becauſe b immediately follows a^{m-4}; thoſe which immediately after ſome Power of a have c, I call Products of the ſecond Claſſis, ſo $a^{m-3}\,c\,c\,d$, is a Product of the ſecond Claſſis; thoſe which immediately after ſome Power of a have d, I call Products of the third Claſſis; and ſo of the reſt.

This

This being done, I multiply all the Products belonging to $z^{m+\overline{r-1}}$ (which precedes immediately z^{m+r}) by b, and divide them all by a; 2°. I multiply by c, and divide by a, all the Products belonging to $z^{m+\overline{r-2}}$, except those of the first Classis; 3°. I multiply by d and divide by a, all the Products belonging to $z^{m+\overline{r-3}}$, except those of the first and second Classis: 4°. I multiply by e and divide by a, all the Terms belonging to $z^{m+\overline{r-4}}$, except those of the first, second, and third Classis, and so on, till I meet twice with the same Term. Lastly, I add to all these Terms the Product of $a^{\overline{m-1}}$ into the Letter, whose Exponent is $r+1$.

Here I must take notice, that by the Exponent of a Letter, I mean the Number which expresses what Place the Letter has in the Alphabet: So 3 is the Exponent of the Letter c, because the Letter c is the 3d in the Alphabet.

It is evident by this Rule, you may easily find all the Products belonging to the several Powers of z, if you have but the Product belonging to z^m, viz. a^m.

To find the Unciæ which ought to be prefixed to every Product, I consider the Sum of Units contained in the Indices of the Letters which compose it (the Index of a excepted) I write as many Terms of the Series $m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}$, &c. as there are Units in the Sum of these Indices; this Series is to be the Numerator of a Fraction whose Denominator in the Product of the several Series $1\times2\times3\times4\times5$, *&c.* $1\times2\times3\times4\times5$, *&c.* $1\times2\times3\times4\times5\times6$, *&c.* the first of which contains as many Terms as there are Units in the Index of b, the second as many as there are Units in the Index of c, the third as many as there are Units in the Index of d, the fourth as many as there are Units in the Index of e, &c.

Demonstration.] To raise the Series $az+bzz+cz^3+dz^4$, &c. to any Power whatsoever, write so many Series equal to it, as there are Units in the Index of the Power demanded. Now it is evident that when these Series are also multiplied, there are several Products in which there is the same Power of z; thus if the Series $az+bzz+cz^3+dz^4$, &c. is raised to its Cube, you have the Products b^3z^6, $abcz$, $aadz^6$, in which you find the same Power z^6. Therefore let us consider what is the Condition that can make some Products to contain the same Power of z; the first thing that will appear in relation to it, is, that in any Product whatsoever, the Index of z is the Sum of the particular Indices of z in the multiplying Terms (this follows from the Nature of the Indices): thus b^3z^6 is the Product of bz^2, bz^2, bz^2, and the Sum of the Indices in the multiplying Terms, is $2+2+2=6$; $abcz^6$ is the Product of az, bzz, cz^3, and the Sum of the Indices of z in the multiplying Terms is $1+2+3=6$; $aadz^6$ is the Product of az, az, dz^4, and the Sum of the Indices of z in the multiplying Terms is $1+1+4=6$. The next thing that appears, is, that the Index of z in the multiplying Terms is the same with the Exponent of the Letter to which z is joined: From which two Considerations it follows, that, to have all the Products belonging to a certain Power of z, you must find all the Products where the Sum of the Exponents of the Letters which compose them, shall always be the same with the Index of that Power. Now this is the Method I use, to find easily all the Products belonging to the same Power of z, let $m+r$ be the Index of that Power, I consider that the Sum of the Exponents of the Letters which compose the Products, must exceed by 1, those which belong to z^{m+r-1}: Now because the Excess of the Exponent of the Letter b, above the Exponent of the Letter a, is 1, it follows, that if each of the Products belonging to z^{m+r-1} is multiplied by b, and divided by a, you will have Products, the Sum of whose Exponents will be $m+r$; likewise the Sum of the Exponent of the Letters which compose the Products belonging to z^{m+r} exceeds by 2 the Sum of the Exponents of the Letters which compose the Products belonging to z^{m+r-1}: Now because the Exponent of the Letter a is less by 2 than the Exponent of the Letter c, it follows, that if each Product belonging to z^{m+r-1} is multiplied by c, and divided by a, you will have other Products, the Sum of whose Exponents is still $m+r$: Now if all the Products belonging to z^{m+r-2} were multiplied by c and divided by a, you would have some Products that would be the same as some of them found before; therefore you must except out of them those that I have called Products of the first Classis. What I have said shews why all the Products belonging to z^{m+r-3}, except those of the first and second Classis, must be multiplied by d, and divided by a. Lastly, you see the Reason why to all these Products is added the Product of a^{m-1} by the Letter whose Exponent is $r+1$; 'Tis because the Sum of the Exponents is still $m+r$.

As for what relates to the Unciæ; observe, that when you multiply $az+bzz+cz^3+dz^4$, &c. by $az+bzz+cz^3+dz^4$, &c. each Letter, a, b, c, d, &c. of the second Series, is multiplied by each of the Letters a, b, c, d, &c. of the first Series. Thus the Letter a of the second Series is multiplied by the Letter b of the first Series, and the Letter b of the second Series is multiplied by the Letter a of the first; therefore you may have the two Planes, ab, ab, or $2ab$; for the same Reason you have $2ac$, $2ad$, &c. Therefore you must prefix to each Plane of those that compose the Square of the Infinite Series $az+bz+cz^3$, &c. the Number which expresses how many Ways the Letters of each Plane may be changed; likewise if you multiply the Product of the two preceding Series by $az+bzz+cz^3$, each Letter, a, b, c, d, of the third Series, is multiplied by each of the Planes formed by the Product of the first and second Series: Thus the Letter a is multiplied by the Planes bc and cb; the Letter b is multiplied by ac and ca; the Letter c is multiplied by ab and ba; therefore you have the six Solids, abc, acb, bac, bca, cab, cba, or six abc: Therefore you must prefix to each Solid whereof the Cube of the Infinite Series is composed, the Number which expresses how many Ways the Letters of each Solid may be changed; and, generally, you must prefix to any Product, whereof any Power of the Infinite Series $az+bzz+cz^3$, &c. is composed, the Number which expresses how many ways the Letters of each Product may be changed.

Now

Now to find how many Ways the Letters of any Product, for inſtance, $a^{m-n}\, b^{h}\, c^{p}\, d^{r}$, may be changed; this is the Rule which is commonly given: Write as many Terms of the Series $1 \times 2 \times 3 \times 4 \times 5$, *&c.* as there are Units in the Sum of the Indices, *viz.* $m - n + h + p + r$; let this Series be the Numerator of a Fraction, whoſe Denominator ſhall be the Product of the Series, $1 \times 2 \times 3 \times 4 \times 5$, *&c.* $1 \times 2 \times 3 \times 4 \times 5$, *&c.* $1 \times 2 \times 3 \times 4 \times 5 \times 6$, *&c.* $1 \times 2 \times 3 \times 4 \times 5$, *&c.* whereof the firſt is to contain as many Terms as there are Units in the firſt Index $m - n$; the ſecond as many as there are Units in the ſecond Index h; the third as many as there are Units in the third Index p; the fourth as many as there are Units in the fourth Index r. But the Numerator and Denominator of this Fraction have a common Diviſor, *viz.* the Series of $1 \times 2 \times 3 \times 4 \times 5$, *&c.* continued to ſo many Terms as there are Units in the firſt Index $m - n$; therefore let both this Numerator and Denominator be divided by this common Diviſor, then this new Numerator will begin with $m - n + 1$, whereas the other began with 1, and will contain ſo many Terms as there are Units in $h + p + r$, that is, ſo many as there are Units in the Sum of all the Indices excepting the firſt: As for the new Denominator, it will be the Product of three Series only; that is, of ſo many as there are Indices, excepting the firſt. But if it happens withal, that n be equal to $h + p + r$, as it always happens in our *Theorem*, then the Numerator beginning by $m - n + 1$, and being continued to ſo many Terms as there are Units in $h + p + r$ or n, the laſt Term will be m neceſſarily: So if you invert the Series, and make that the firſt Term which was the laſt, the Numerator will be $m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}$, *&c.* continued to ſo many Terms as there are Units in the Sum of the Indices of each Product, excepting the firſt Index. And whatever is here ſaid of Powers whoſe Index is an Integer, may be adapted to Roots or Powers whoſe Index is a Fraction.

XXII. *Theorem.*] If $az + bzz + cz^3 + dz^4 + ez^5 + fz^6$, *&c.* $= gy + hyy + iy^3 + ky^4 + ly^5 + my^6$, *&c.* then will z be $= \frac{g}{a}y + \frac{h - bAA}{a}y^2 + \frac{i - 2bAB - cA^3}{a}y^3 + \frac{k - bBB - 2bAC - 3cAAB - dA^4}{a}y^4 + \frac{l - 2bBC - 2bAD - 3cABB - 3cAAC - 4dA^3B - eA^5}{a}y^5 + \frac{m - 2bBD - bCC - 2bAE - cB^3 - 6cABC - 3cAAD - 6dAABB - 4dAC - 5eA^4B - fA^6}{a}y^6$, &c. *The Extraction of a Root of an Infinite Equation; by Mr. Abr. de Moivre.* N. 240. *p.* 190. May, An. 1698.

For the underſtanding of this Series, and in order to continue it as far as we pleaſe, it is to be obſerved, 1. That every Capital Letter is equal to the Coefficient of each preceding Term; thus the Letter B is equal to the Coeffi-

cient

cient $\frac{h - bAA}{a}$. 2. That the Denominator of each Coefficient is always a. 3. That the firſt Member of each Numerator is always a Coefficient of the Series $gy + hyy + iy^3$, &c. *viz.* the firſt Numerator begins with the firſt Coefficient g, the ſecond Numerator with the ſecond Coefficient h, and ſo on. 4. That in every Member after the firſt, the Sum of the Exponents of the Capital Letters is always equal to the Index of the Power to which this Member belongs: Thus, conſidering the Coefficient $\frac{k - bBB - 2bAC - 3cAAB - dA^4}{a}$, which belongs to the Power^4, we ſhall ſee that in every Member bBB, $2bAC$, $3cAAB$, dA^4, the Sum of the Exponents of the Capital Letters is 4, (where I muſt take notice, that by the Exponent of a Letter, I mean the Number which expreſſes what place it has in the Alphabet; thus 4 is the Exponent of the Letter D) hence I derive this Rule for finding the Capital Letters of all the Members that belong to any Power; Combine the Capital Letters as often as you can make the Sum of their Exponents equal to the Index of the Power to which they belong. 5. That the Exponents of the ſmall Letters, which are written before the Capitals, expreſs how many Capitals there are in each Member. 6. That the Numerical Figures or Unciæ that occur in theſe Members, expreſs the Number of Permutations which the Capital Letters of every Member are capable of.

For the *Demonſtration* of this, Suppoſe $z = Ay + Byy + Cy^3 + Dy^4$, *&c.* Subſtitute this Series in the room of z, and the Powers of this Series in the room of the Powers of z, there will ariſe a new Series: Then take the Coefficients which belong to the ſeveral Powers of y in this new Series, and make them equal to the correſponding Coefficients of the Series $gy + hyy + iy^3$, &c. and the Coefficients A, B, C, D, *&c.* will be found ſuch as I have determined them.

But if any one deſires to be ſatisfied, that the Law by which the Coefficients are formed will always hold, I'll deſire them to have recourſe to the *Theorem* I have given for *Raiſing an Infinite Series to any Power*, or extracting any Root of the ſame; for if they make uſe of it, for taking ſucceſſively the Powers of $Ay + Byy + Cy^3$, &c. they will ſee that it muſt of neceſſity be ſo. I might have made the *Theorem* I give here much more general than it is; for I might have ſuppoſed, $az^m + bz^{m+1} + cz^{m+2} + dz^{m+3}$, *&c.* $= gy^m + hy^{m+1} + iy^{m+2}$, *&c.* then all the Powers of the Series $Ay + Byy + Cy^3$, *&c.* deſigned by the univerſal Indices, muſt have been taken ſucceſſively; but thoſe who will pleaſe to try this, may eaſily do it, by means of the *Theorem* for raiſing an Infinite Series to any Power, *&c.* *Sect.* XXI.

This *Theorem* may be applied to what is called the Reverſion of Series; ſuch as finding the Number from its Logarithm given; the Sign from the Arch; the Ordinate of an Ellipſe from an Area given to be cut from any Point in the Axis: But to make a particular Application of it, I will ſuppoſe we have this Problem to ſolve; *viz. The Chord of an Arc being given, to find*

the

the Chord of another Arc, that ſhall be to the firſt as n *to* 1. Let y be the Chord given, z the Chord required; now the Arc belonging to the Chord y is $y + \frac{y^3}{6dd} + \frac{3y^5}{40d^4} + \frac{5y^7}{112d^6}$, &c. And the Arc belonging to the Chord z is $z + \frac{z^3}{6dd} + \frac{3z^5}{40d^4} + \frac{5z^7}{112d^6}$, &c. The firſt of theſe Arcs is to the ſecond as 1 to n; therefore multiplying the Extremes and Means together, we ſhall have this Equation: $z + \frac{z^3}{6dd} + \frac{3z^5}{40d^4} + \frac{5z^7}{112d^6}$, &c. $= ny + \frac{ny^3}{6dd} + \frac{3ny^5}{40d^4} + \frac{5ny^7}{112d^6}$, &c.

Compare theſe two Series with the two Series of the Theorem, and you will find $a = 1$, $b = 0$, $c = \frac{1}{6dd}$, $d = 0$, $e = \frac{3}{40d^4}$, $f = 0$, &c. $g = n$, $h = 0$, $i = \frac{n}{6dd}$, $k = 0$, $l = \frac{3n}{40d^4}$, $m = 0$, &c. Hence z will be $= ny + \frac{n - n^3}{6dd}y^3$, *&c.* or $ny + \frac{1 - nn}{2 \times 3dd}yyA$, *&c.* ſuppoſing A to denote the whole preceding Term; which will be the ſame Series as Mr. *Newton* has firſt found.

By the ſame Method this general *Problem* may be ſolved: *The Abſciſs correſponding to a certain Area in any Curve being given, to find the Abſciſs whoſe correſponding Area ſhall be to the firſt in a given Ratio.*

The Logarithmic Series might alſo be found without borrowing any other Idea, than that Logarithms are the Indices of Powers: Let the Number, whoſe Logarithm we enquire, be $1 + z$; ſuppoſe its Log. to be $az + bzz + cz^3$, &c. Let there be another Number $1 + y$; therefore its Logarithm will be $ay + byy + cy^3$, &c. Now if $1 + z = \overline{1+y}^n$, it follows that $az + bzz + cz^3$, &c. : $ay + byy + cy^3$, &c. :: n : 1; that is, $az + bzz + cz^3$, &c. $= nay + nbyy + ncy^3$, &c. therefore we may find a Value of z expreſſed by the Powers of y. Again, ſince $1 + z = \overline{1+y}^n$, therefore $z = \overline{1+y}^n - 1$; that is, $z = ny + \frac{n}{1} \times \frac{n-1}{2}yy + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3$, &c. Therefore z is doubly expreſſed by the Powers of y. Compare theſe two Values together, and the Coefficients a, b, c, &c. will be determined, except the firſt, a, which may be taken at pleaſure, and gives accordingly all the different Species of Logarithms.

XXIII. You need be no longer in Concern, how the infiniteſimal Differences of Infiniteſimals are to be explained. For as you will eaſily grant, that any *Doctrine of Exhauſtions; by* Dr. Wallis. N. 255. *p.* 282. Aug. An. 1699.

any Multiple of Nothing is ſtill Nothing; I ſhall allow with the ſame Eaſe, that you may ſafely neglect infiniteſimal Differences drawn into Infiniteſimals. But though this may be done ſafely, yet it ought to be done cautiouſly. For in any kind of Quantities, thoſe Things which differ by leſs than a given or aſſignable Difference, are to be conſidered as equal. Upon this the whole Doctrine of Exhauſtions is founded, which is neceſſary both to the Antients and Moderns.

The Approximation of the Ancients in the Extracting of Roots, improv'd by Dr. Wallis. N. 215. *p.* 2. Jan. An. 1695.

XXIV. It is agreed by all, that if a Number propoſed be not a true Square, it is in vain to hope for a juſt Quadratic Root thereof, explicable by rational Numbers, Integers or Fracted. And therefore in ſuch Caſes we muſt content ourſelves with Approximations (ſomewhat near the Truth) without pretending to Accuracy.

And ſo, for the Cubic Root of what is not a perfect Cube. And the like for ſuperior Powers.

Now the Ancients (being aware of this) had their Methods of Approximation, which though ſcarce applied by them beyond the Quadratic, or perhaps the Cubic Root, yet are equally practicable (by due Adjuſtments) to the ſuperior Powers alſo.

I ſhall begin with the Square Root: For which the ancient Method is to this Purpoſe.

From the propoſed Non-quadrate (ſuppoſe N) ſubtract (in the uſual manner) the greateſt Square in Integers therein contained (ſuppoſe Aq.) The Remainder (ſuppoſe $B = 2AE + Eq$) is to be the Numerator of a Fraction, for deſigning the near Value of E (the remaining Part of the Root ſought $A + E = \surd N$) whoſe Denominator or Diviſor is to be 2 A (the double Root of the ſubtracted Square) or $2A + 1$ (that double Root increaſed by 1) the true Value falling between theſe two; ſometime the one, ſometime the other, being neareſt to the true Value. But (for avoiding of Negative Numbers) the latter is commonly directed.

This Method Monſieur *De Lagny* affirms to be more than 200 Years old. And it is ſo; for I find it in *Lucas Pacciolus* (otherwiſe called *Lucas de Burgo*, or *de Burgo Sancti Sepulchri)* printed at *Venice* in the Year 1494, (if not even ſooner than ſo, for I find there have been ſeveral Editions of it.) And how much older than ſo, I cannot tell: For he doth not deliver it as a new Invention of his own, but as a received Practice, and derived from the *Moors* or *Arabs*, from whom they had their *Algoriſm*, or Practice of Arithmetick by the ten numeral Figures now in Uſe.

And it is continued down hitherto in Books of practical Arithmetick in all Languages, which teach the Extraction of the Square Root, and (therein) this Method of Approximation, in caſe of a Non-quadrate.

The true Ground of the Rule is this: Aq being (by Conſtruction) the greateſt Integer Square contained in N, 'tis evident that E muſt be leſs than 1; (otherwiſe not Aq, but the Square of $A + 1$, or ſome greater than ſo, would be the greateſt Integer Square contained in N.) Now if the Remainder $B = 2AE + Eq$ be divided by 2 A, the Reſult will be too great for E, (the

(the Divisor being too little; for it should be $2A+E$, to make the Quotient E.) But if (to rectify this) we diminish the Quotient by increasing the Divisor, adding 1 to it, it then becomes too little; because the Divisor is now too big. For (E being less than 1) $2A+1$ is more than $2A+E$; and therefore too big.

As for Instance: If the Non-quadrate proposed be $N=5$, the greatest Integer Square therein contained is $Aq=4$ (the Square of $A=2$:) which being subtracted, leaves $N-Aq=5-4=1=B=2AE+Eq$. Which divided by $2A=4$ gives $\frac{1}{4}$; but divided by $2A+1=4+1=5$, gives $\frac{1}{5}$. That too great, and this too little, for E. And therefore the true Root ($A+E=\sqrt{N}$) is less than $2\frac{1}{4}=2.25$, but greater than $2\frac{1}{5}=2.2$: And this was anciently thought an Approach near enough.

If this Approach be not now thought near enough, the same Process may be again repeated; and that as oft as is thought necessary.

Take now for A, $2\frac{1}{5}=2.2$, whose Square is $4.84=Aq$, (now considered as an Integer in the second Place of Decimal Parts.) This subtracted from 5.00, (or, which is the same, 0.84, the Excess of this Square above the former, from 1, which was then the Remainder) leaves a new Remainder $B+0.16$: which divided by $2A=4.4$, gives $\frac{0.16}{4.40}=\frac{2}{55}=0.03636+$, too much. But divided by $2A+1=4.5$, it gives $\frac{0.16}{4.50}=\frac{8}{225}=0.03555+$, too little. The true Value (between these two) being 2.236 *proxime*, whose Square is 4.999696.

If this be not thought near enough, subtract the Square from 5.000000: the Remainder $B=0.000304$, divided by $2A=4.472$, or by $2A+1=4.473$, gives (either way) $0.000068-$; which added to $A=2.236$, makes $2.236068-$, somewhat too big; but $2.236067+$ would be much more too little.

Which gives us the Square Root of 5, adjusted to the sixth Place of Decimal Parts, at three Steps. And by the same Method, if it be thought needful, we may proceed further.

For the Cubic Root the Rule is this:

From the Non-Cubic proposed, (suppose N) subtract the greatest Cube in Integers therein contained, (suppose Ac:) the Remainder (suppose $B=3AqE+3AEq+Ec$,) is to be the Numerator of a Fraction for designing the Value of E, (the remaining Part of the Root sought $A+E=\sqrt{c}N$.) To this Numerator, if (for the Denominator or Divisor) we subjoin $3Aq$, the Result will certainly be too great for E, because the Divisor is too little: (For it should be $3Aq+3AE+Eq$, to give the true Value of E.) If for the Divisor we take $3Aq+3A+1$, it will certainly be too little, because the Divisor is too great. (For E by Construction is less than 1.) It must therefore (between these Limits) be more than this latter. And therefore this latter Result being added to A, will give a Root whose Cube may be subtracted from the Non-Cubic proposed, in order to another Step.

But if, for the Divisor, we take 3 A q + 3 A, (or even less than so) the Result may be too great; or (in case B be small) it may be too little, and oft is so. Which comes to pass from hence; because E (by Construction) is less than 1; and therefore 3 A E less than 3 A; and perhaps so much as that the Addition of E q will not redress it. And when it so happens 3 A q + 3 A is a better Divisor than 3 A q + 3 A + 1, (or even somewhat less than either.) But because it doth not always so happen (though for the most part it doth) the Rule doth rather direct the other; as which doth certainly give a Root less than the true Value, whose Cube may always be subtracted from the Non-Cubic proposed: the Design being to have such a Cube as (being subtracted) may leave another B, to be ordered in like manner for a new Approach.

But, for the most part, 3 A q may be safely taken for the Divisor. For tho' the Result will then be somewhat too big, yet the Excess may be so small as to be neglected; or, at least, we may thence easily judge what Number (somewhat less than it) may be safely taken. And if we chance to take it somewhat too big, the Inconvenience will be but this, that B for the next Step will be a Negative. Of which Case we shall speak anon.

Thus for Instance: If the Non-Cube proposed be 9 = N: The greatest Integer Cube therein contained is 8 = A c, (whose Cubic Root is A = 2.) Which Cube subtracted, leaves 9 − 8 = 1 = B = 3 A q E + 3 A E q + E c. This divided by 3 A q = 12, gives $\frac{1}{12}$ = 0.08333 +, too big for E. But the same divided by 3 A q + 3 A + 1 = 12 + 6 + 1 = 19, gives $\frac{1}{19}$ = 0.05263 +, too little. Or if but by 3 A q + 3 A = 12 + 6 = 18, it gives $\frac{1}{18} = \frac{1}{18}$ = 0. 05555 +, yet too little. For the Cube of A + 0. 06, = 2. 06, is but 8. 742 —, which is short of 9; and so much short of it, that we may safely take 2. 07 as not too big: Or perhaps 2. 08, which upon Trial will be found not too big; for the Cube of 2. 08, is but 8. 998912.

If this first Step be not near enough, this Cube subtracted from 9. 000000, leaves a new B = 0. 001088, which divided by 3 A q = 12. 9796, gives 0.000084 —; which will be somewhat too big, but not much. (For E is now so small, as that 3 A E may be safely neglected, and E q much more.) So that if to 2. 08, we add 0. 000084 —, the Result 2. 080084 will be too big; but 2. 080083 will be more too little, (as will appear if we take the Cube of each.) So that either of them, at the second Step, gives the true Root within an Unit in the sixth Place of decimal Parts. But when I say, taking the Cube of each, (which I do that the Thing may be more clearly apprehended) it is not necessary that we trouble ourselves with the whole Cube. For A c being already subtracted, for finding B = 3 A q E + 3 A E q + E c, we have no more to try, but whether 3 A q E + 3 A E q + E c, be greater or less than B, according as we take 0. 000084, or 0. 000083, for E.

Which may conveniently be done in this manner: Take 3 A + E, and multiply this by E, (or E by it) so have we 3 A E + E q. To this add 3 A q, and multiply the whole by E, (so have we 3 A q E + 3 A E q + E c) to see whether this be greater or less than B.

That is, in the present Case, if we take E = 0. 000084, and add to this 3 A = 6. 24, then is 6. 240084 = 3 A + E. This multiplied by E = 0. 000084, 0.000084, is $3\,AE + Eq = 0.000524+$. To which if we add $3\,Aq = 12.9792$, it is $3\,Aq + 3\,AE + Eq = 12.979724$. Which multiplied again by $E = 0.000084$, is $0.0010902 + = 3\,AqE + 3\,AEq + Ec$, which is more than $B = 0.001088$.

But if we take $E = 0.000083$, and proceed as before, we ſhall have $3\,AqE + 3\,AEq + Ec = 0.001077+$, which is leſs than $B = 0.001088$. And therefore (if we ſubtract that from this) the Remainder, 0.000011, will be another B for the next Step, if we pleaſe to proceed farther.

Hitherto I have purſued the Method moſt affected by the Ancients, in ſeeking a Square or Cube (and the like of other Powers) always leſs than the juſt Value, that it might be ſubtracted from the Number propoſed, leaving B a poſitive Remainder; thereby avoiding Negative Numbers.

But ſince the Arithmetic of Negatives is now ſo well underſtood, it may in this (and other Operations of like nature) be adviſeable to take the neareſt, whether it be greater or leſs than the juſt Value.

According to this Notion, for the Square Root of 5, I would ſay it is ($2+$) ſomewhat more than 2; and enquire how much more. But for the Square Root of 8, I would ſay, it is ($3-$) ſomewhat leſs than 3; and enquire how much leſs: taking (in both Caſes) that which is neareſt to the juſt Value.

Thus in the Cubic Root before us; I would take for E (in the laſt Enquiry) $0.000084-$ (where for the next Step we have $B = -0.000002$) rather than $0.000083+$ (where for the next Step we ſhould have $B = +0.000011$.) In the latter Caſe, we are to divide $B = +0.000011$, by $3\,Aq = 12.980236-$, to find (by the Quotient) how much is to be added to 0.000083. In the other Caſe, we are to divide $B = +0.000002$, by $3\,Aq = 12.98028$, to find (by the Quotient) what is to be abated of 0.000084; ſo have we $\frac{0.000011}{12.980236} = 0.00000085+$ to be added to 6.240083: Or $\frac{0.000002}{12.980248} = 0.00000015+$ to be abated of 6.240084; (Or it may ſuffice in either to divide by $12.98+$, or even by $13-$, without being incumbered with a long Diviſor) either of which gives us, for the Root ſought, 2.08008385 *proxime*; true (at the third Step) to the eighth Place of decimal Parts. And if this be not near enough, the Cube of this, compared with the Number propoſed, will give us another B for the next Step: And ſo onwards as far as we pleaſe.

Now what is ſaid of the Cube, is eaſily applicable to the higher Powers.

I ſhall omit that of the Biquadrate; becauſe here perhaps it may be thought moſt adviſeable to extract the Square Root of the Number propoſed, and then the Square Root of that Root. But if we would do it at once, we are from N (the Number propoſed, being not a Biquadrate) to ſubtract $A\,qq$ (the greateſt Biquadrate contained in it) to find the Remainder $B = 4\,AcE + 6\,AqEq + 4\,AEc + Eqq$. Which Remainder, if we divide by $4\,Ac$, the Quotient will certainly be too big for E, (though perhaps not much:) If by $4\,Ac + 6\,Aq + 4\,A + 1$, it will certainly be too little (for Reaſons before-mentioned.) And we are to use our Discretion in taking some intermediate Number. And if we chance not to hit on the nearest, the Inconvenience will be but this, that our Leap will not be so great as otherwise it might be. Which will be rectified by another B at the next Step.

For the Sursolid (of five Dimensions) we are, from N (the Number proposed, being not a perfect Sursolid) to subtract A qc (the greatest Sursolid therein contained) to find the Remainder B $= 5$ A qq E $+ 10$ A c E $q + 10$ A q E $c + 5$ A E $qq +$ E qc. Which (as before) if we divide by 5 A qq, the Result will be somewhat too big, (because the Divisor is too little:) If by 5 A $qq + 10$ A $c + 10$ A $q + 5$ A $+ 1$, the Result will certainly be less than the true E. The just Value of E being somewhat between these two, where we are to use our Discretion what intermediate Number to take. Which according as it proves too great or too little, is to be rectified at the next Step.

But for the most part it will be safe enough (and least trouble) to divide by 5 A qq, which gives a Quotient somewhat too big; which we may either rectify at Discretion, by taking a Number somewhat less, or proceed to another B, (affirmative or negative, as the Case shall require) and so onward to what Exactness we please. Which is, for Substance, in a manner co-incident with Mr. *Raphson*'s Method, even for affected Equations.

Thus, in the present Case: If the Number proposed be N $= 33$, then is A $qc = 32$, and B $= 33 - 32 = 1 = 5$ A qq E $+ 10$ A c E $q + 10$ A q E $c + 5$ A E $qq +$ E qc. Which if we divide by 5 A $qq = 5 \times 16 = 80$, the Result $\frac{1}{80} = 0.0125$, is somewhat too big for E, but not much. And if we examine it, by taking the Sursolid of 2.0125, or of $2\frac{1}{80}$, we shall find a Negative B (for the next Step), but not very considerable. Or if we think it considerable, we may proceed farther to another Step, or more than so.

The like Method may be applied (and with more Advantage) in the higher Powers, according as the Composition of each Power requires.

And the same Method may be of Use (with good Advantage) in long Numbers (if duly applied) even before we come to the Place of Units; for the same will equally hold there also. Which the Reader may easily apprehend, without a long Discourse upon it.

The Proportion of infinite Quantities; by Mr. E. Halley. n. 195. p. 556.

XXV. The very Idea of Magnitudes infinitely great, or such as exceed any assignable Quantity, does include a Negation of Limits: yet if we nearly examine this Notion, we shall find that such Magnitudes are not equal amongst themselves, but that there are really besides infinite Length, and infinite Area, three several Sorts of infinite Solidity: all of which are *Quantitates sui generis*; and that those of each Species are in given Proportions.

Infinite Length, or a Line infinitely long, is to be considered either as beginning at a Point, and so infinitely extended one Way, or else both Ways from the same Point; in which Case the one, which is a beginning Infinity, is the one half of the whole, which is the Sum of the beginning and ceasing Infinity; or, as I may say, of Infinity *à parte ante* and *à parte post*, which is analogous to Eternity in Time or Duration, in which there is always as much

much to follow as is paſt, from any Point or Moment of Time: Nor doth the Addition or Subduction of finite Length, or Space of Time, alter the Caſe either in Infinity or Eternity, ſince both the one or the other cannot be any Part of the Whole.

As to Infinite Surface or Area, any Right Line infinitely extended both Ways on an infinite Plane, does divide that infinite Plane into equal Parts, the one to the right, and the other to the left of the ſaid Line; but if from any Point in ſuch a Plane, two right Lines be infinitely extended, ſo as to make an Angle, the infinite Area, intercepted between thoſe infinite right Lines, is to the whole infinite Plane, as the Arch of a Circle on the Point of Concourſe of thoſe Lines as a Center, intercepted between the ſaid Lines, is to the Circumference of the Circle; or as the Degrees of the Angle to the 360 Degrees of a Circle. For Example: Two right Lines meeting at a right Angle do include, on an infinite Plane, a quarter Part of the whole infinite Area of ſuch a Plane.

But if ſo be two parallel infinite Lines be ſuppoſed drawn on ſuch an infinite Plane, the Area intercepted between them will be likewiſe infinite; but at the ſame time will be infinitely leſs than that Space, which is intercepted between two infinite Lines that are inclined, tho' with never ſo ſmall an Angle; for that in the one Caſe, the given finite Diſtance of the parallel Lines diminiſhes the Infinity in one Degree of Dimenſion; whereas in a Sector, there is Infinity in both Dimenſions: and conſequently the Quantities are the one infinitely greater than the other, and there is no Proportion between them.

From the ſame Conſideration ariſe the three ſeveral Species of infinite Space or Solidity; for a Parallelopipid, or a Cylinder infinitely long, is greater than any finite Magnitude, how great ſoever; and all ſuch Solids ſuppoſed to be formed on given Baſes, are as thoſe Baſes in proportion to one another. But if two of theſe three Dimenſions are wanting, as in the Space intercepted between two parallel Planes infinitely extended, and at a finite Diſtance; or with infinite Length and Breadth, with a finite Thickneſs, all ſuch Solids ſhall be as the given finite Diſtances one to another; but theſe Quantities, tho' infinitely greater than the other, are yet infinitely leſs than any of thoſe wherein all the three Dimenſions are infinite. Such are the Spaces intercepted between two inclined Planes infinitely extended; the Space intercepted by the Surface of a Cone, or the Sides of a Pyramid, likewiſe infinitely continued, *&c.* of all which notwithſtanding, the Proportions one to another, and to the τὸ πᾶν or vaſt Abyſs of infinite Space (wherein is the *Locus* of all Things that are or can be; or to the Solid of infinite Length, Breadth, and Thickneſs taken all manner of ways) are eaſily aſſignable. For the Space between two Planes is to the whole, as the Angle of thoſe Planes to the 360 Degrees of the Circle. As for Cones and Pyramids, they are as the ſpherical Surface intercepted by them is to the Surface of the Sphere, and therefore Cones are as the verſed Sines of half their Angles to the Diameter of the Circle: Theſe three Sorts of infinite Quantity are analogous to a Line, Surface,

face, and Solid, and after the same manner cannot be compared, or have no Proportion the one to the other.

Infinitely-infinite Fractions; by D. R. Wood. Ph. Col. n. 3. p. 45.

XXVI. Infinitely-infinite Fractions, or all the Powers of the Fractions whose Numerator is 1, are all of them together equal to (1) an Unit.

$$\overbrace{\begin{array}{rcccccl}
 & \text{A.} & \text{R.} & \text{q.} & \text{c.} & \text{qq.} & \\
1 = & \frac{1}{1} = & \frac{1}{2} : + & \frac{1}{4} + & \frac{1}{8} + & \frac{1}{16} & \&c. \\
 & \frac{1}{2} = & \frac{1}{3} : + & \frac{1}{9} + & \frac{1}{27} + & \frac{1}{81} & \&c. \\
 & \frac{1}{3} = & \frac{1}{4} : + & \frac{1}{16} + & \frac{1}{64} + & \frac{1}{256} & \&c. \\
 & \frac{1}{4} = & \frac{1}{5} : + & \frac{1}{25} + & \frac{1}{125} + & \frac{1}{625} & \&c. \\
 & \frac{1}{5} = & \frac{1}{6} : + & \frac{1}{36} + & \frac{1}{216} + & \frac{1}{1296} & \&c. \\
 & \&c. & \&c. & & & &
\end{array}}^{\text{P.}}
\quad \text{Or thus:} \quad
\begin{array}{c|c|c|}
 & \text{1 anor. +} & \\
\frac{1}{1\times1} = & \frac{1}{2} + & \frac{1}{1\times2} = \\
\frac{1}{2\times1} = & \frac{1}{3} + & \frac{1}{2\times3} = \\
\frac{1}{3\times1} = & \frac{1}{4} + & \frac{1}{3\times4} = \\
\frac{1}{4\times1} = & \frac{1}{5} + & \frac{1}{4\times5} = \\
\frac{1}{5\times1} = & \frac{1}{6} + & \frac{1}{5\times6} =
\end{array}$$

$$\begin{array}{c|c|c|c|c}
\text{2 anor. +} & & \text{3 anor. +} & & \text{4 anor. +} \\
\frac{1}{2^2} + & \frac{1}{1\times2^2} = & \frac{1}{2^3} + & \frac{1}{1\times2^3} = & \frac{1}{2^4} + \&c. \\
\frac{1}{3^2} + & \frac{1}{2\times3^2} = & \frac{1}{3^3} + & \frac{1}{2\times3^3} = & \frac{1}{3^4} + \&c. \\
\frac{1}{4^2} + & \frac{1}{3\times4^2} = & \frac{1}{4^3} + & \frac{1}{3\times4^3} = & \frac{1}{4^4} + \&c. \\
\frac{1}{5^2} + & \frac{1}{4\times5^2} = & \frac{1}{5^3} + & \frac{1}{4\times5^3} = & \frac{1}{5^4} + \&c. \\
\frac{1}{6^2} + & \frac{1}{5\times6^2} = & \frac{1}{6^3} + & \frac{1}{5\times6^3} = & \frac{1}{6^4} + \&c.
\end{array}$$

A. Is a File or Row of absolute Numbers, or rather of all the Fractions, whose Numerator is 1; which Row is supposed to be continued *in infinitum* (downwards.)

R. Is

R. Is another File or Row of all the Roots (whose Numerator is 1) of all the Powers of such Fractions, supposed likewise to be continued *in infinitum* (downwards).

P. Are all the respective Powers of such Fractions, (*viz.* Squares, Cubes, *&c.*) or so many Ranks of Geometrical Proportionals, supposed to be continued *in infinitum*, both to the Right-hand, and also downwards.

Lemma.] *Each of the said Ranks of Powers, together with their respective Roots, is equal to each of the several Numbers under* A *respectively.*

Demonstration.] If from the Line ab you take (for instance) $\frac{1}{4}$ part towards *Fig.* 54. a, suppose ac; and also from the other End of the same Line ab, you take two such Parts (or $\frac{2}{4}$ Parts) towards b, suppose bd, (*viz.* a Number of Parts less by two than the whole Line ab was first supposed to be divided into) there will remain the Line $cd = ac = \frac{1}{4}$ of ab. Then again, if from cd you take $\frac{1}{4}$ Part thereof towards a, suppose ce, and from the other End $\frac{2}{4}$ Part of the same cd, suppose df, there will remain only $ef = ce = \frac{1}{4}$ of cd. And if you still go on without ceasing, to take on the Side towards a, $\frac{1}{4}$ Part of what was taken last before, and twice as much on the other Side towards b, there shall be found between the two Lines last taken always remaining $\frac{1}{4}$ Part of the Line from which they were taken. From which $\frac{1}{4}$ Part there may still after the same manner be supposed to be taken two other such Lines. But if this be supposed to be done infinite Times actually, then there will nothing more remain (between), and so the continued Division on either Side will come exactly to the Point g, supposing ag to be $\frac{1}{3}$ of ab, and $bg = 2\,ag$: For, because that which was taken away towards b, was always twice as much as that which was taken away towards a, the total Sum of all the Lines taken away towards b, (which all together do make up the Line bg) must be twice as much as the Line ag, (which is the total Sum of all the Lines taken away towards a) viz. $bg = 2\,ag$; and consequently $bg + ag$ (or the whole Line ab) is equal to $3\,ag$; and therefore $ag = \frac{1}{3}$ of ab. *Q. E. D.*

The like *Construction* and *Demonstration* (*mutatis mutandis*) may be made use of in taking away any other Part of any Quantity, and the like Part again of the first mentioned Part, and so *in infinitum*. The total Sum of all the Parts so taken, or supposed to be taken, shall be equal to any certain Quantity, or Part, or Fraction, whose Denominator shall be less by an Unit than the Denominator of the first mentioned Part; as $\frac{1}{6} = \frac{1}{7} + \frac{1}{49} + \frac{1}{343}$ &c. $\frac{1}{9} = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$ &c. And so, that which the incomparable *Archimedes* (in his Squaring the *Parabola*) has only demonstrated in one particular Case, *viz.* $\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024}$ &c. and that too, not without an huge Apparatus of Preliminary Propositions, amounting to a whole Book, is here universally de-

monstrated

monſtrated in all Caſes (which are infinite) and by a very ſimple and eaſy Method (in *Des Cartes*'s Way) and on one ſingle Page.

Now if each of the ſaid Ranks of Powers, together with their reſpective Roots, be equal to the ſeveral Numbers of Fractions under A; (as is demonſtrated by the *Lemma*) then is A the Sum of them all, or equal to them all; that is to ſay, $R + P = A = R + 1$; for R is the ſame with A wanting $\frac{1}{1}$, or 1, (as appears upon View) or but $(\frac{1}{00})$ one infinite Part bigger. Wherefore $P = 1$, *Q. E. D. viz.* Infinitely-infinite Fractions are equal to Unity, that is, to the leaſt Root that is an Integer.

Corollaries. Hence it is plain,

1. That a Progreſs *ad infinitum* muſt be allowed.
2. And not only to one Infinite, but to ſeveral; or rather to infinite Infinites, or infinitely Infinites.
3. And that this may be done, that is, this Calculation may be performed, by a very finite or bounded Capacity.
4. And that this whole Progreſs, or ſuch infinite Progreſſes, may be caſt up, or collected into one Sum.
5. And into a Sum that is not only not infinite, but ſo ſmall that it is leſs than any Number.

It appears farther,

That of Infinites ſome are equal, and others unequal.

And that one Infinite may be equal to two, three, or ſeveral, either Finites or Infinites.

For 1. The infinite Powers of the firſt Rank are $= \frac{1}{2} = \frac{1}{1 \times 2}$, and alſo equal to all the infinitely-infinite Powers of all the other Ranks.

The infinite Powers of the ſecond Rank are $= \frac{1}{6} = \frac{1}{2 \times 3}$	*viz.* equal to the reſpective mean proportional Number between the Square Numbers reſpectively.
Thoſe of the third Rank are $= \frac{1}{12} = \frac{1}{3 \times 4}$	*e. g.*
Thoſe of the fourth Rank are $= \frac{1}{20} = \frac{1}{4 \times 5}$	4, 6, 9
Thoſe of the fifth Rank are $= \frac{1}{30} = \frac{1}{5 \times 6}$	9, 12, 16
&c. *in infinitum.*	16, 20, 25
	25, 30, 36
	&c. *in infinitum.*

2. The

2. The infinite Powers of the two firſt Ranks are $= \frac{2}{3}$

Thoſe of the three firſt are $= \frac{3}{4}$

Thoſe of the four firſt are $= \frac{4}{5}$

Thoſe of the five firſt are $= \frac{5}{6}$

&c. *in infinitum.*

3. All the Powers of all the infinite Ranks, except the firſt are $= \frac{1}{2}$

All, except the two firſt, are $= \frac{1}{3}$

All, except the three firſt, are $= \frac{1}{4}$

All, except the four firſt, are $= \frac{1}{5}$

&c. *in infinitum.*

The latter *Corollaries* may all appear by ſimple Addition and Subduction; and ſo may many more.

XXVII. 1. That the Numeral Figures now in uſe, with the manner of Computation by them (and the Names of Algoriſm, appropriated to that Way of Computation) came to us from the *Arabs* (but ſomewhat altered, as to the Shape of the Figures, in ſucceeding Ages) and to them from the *Indians*, is generally agreed. But it is not ſo generally agreed, of what Antiquity the Uſe of them, in our *European* Parts, hath been. *The Antiquity of the Numeral Figures; by Dr. J. Wallis. N. 154. p. 399. Dec. An. 1683.*

Jo. Gerard Voſſius (De Scientiis Mathematicis) thinks they came not in Uſe here till about the Year of our Lord 1300; or at the fartheſt, later than the Year 1250. And *P. Mabillon* (*De Re Diplomatica*) tells us, that he hath not found them any where uſed ſooner than the 14th Century. But I think their Uſe in theſe Parts was as old at leaſt as the Times of *Hermannus Contractus*, who lived about the Year of our Lord 1050 (that is, about the middle of the 11th Century,) if not ſo frequently in ordinary Affairs; yet at leaſt in Mathematical Things, and eſpecially in Aſtronomical Tables.

But I do not remember that I have any where ſeen any Monument of them more antient than the Mantle-tree of the Parlour Chimney at the Dwelling-Houſe of Mr. *Will. Richards*, the Rector of *Helmdon* in *Northamptonſhire*. *Fig. 55.*

The Sides of the Chimney, by which the Mantle-tree is ſupported, are of Stone; but the Mantle-tree itſelf is of Wood. It is all over as black as Ink, having by Age and Smoak contracted that Colour. It may yet continue many hundreds of Years; for I did not diſcern in it any Thing either of Worm, or Rottenneſs, or any Tendency to it. The Length of it is five Foot nine Inches; its Breadth or Depth at the Ends, (A B) is 11 ½ Inches, but at the

the Middle, as C D, ſomewhat leſs. It is alſo carved from End to End, and the lower Part of it is abated, as in the Mouldings of other Chimneys. On the Front of the upper Part there is (beginning at the Middle) on three Squares parted from each other by a deep Furrow or Channel, the Date (I ſuppoſe, when it was firſt made) deſcribed partly in Numeral Figures, A° Do^{i} M° 133; on a fourth a Flower, and on a fifth the two Letters W. R. with an Eſcutcheon, repreſenting (I ſuppoſe) the Name of him to whom it did then belong. Both the Letters and the Figures are of an Antique Form, agreeing well enough with that Age. They are not engraved or cut in, but prominent on their ſeveral Squares (by way of *Bas-relief*) the Wood being abated round about them. The o over the A is a round o, but that over the M is a ſquare o.

Hence it appears, that the Uſe of ſuch Figures here in *England*, even on ordinary Occaſions, is at leaſt as ancient as the Year 1133. And I judge it to have been yet ſomewhat ancienter, becauſe the Shape of the Figures, though not come juſt to the Shape which we now uſe, was even then conſiderably varied from the Shape of the *Arabick* Figures; which argues they had then been for ſome Time in uſe; ſuch Change of Shape in Figures and Letters coming on gradually with Time.

It need not move any Scruple at all, that Part of the Numbers is expreſſed by the Numeral Letter M (or the Word *Milleſimo*, of which M° is but a Contraction) while the reſt is expreſſed in Numeral Figures: For the like doth oft occur in old *Manuſcripts*; and ſometimes even at this Day. And it doth rather favour the Simplicity of that Age, (not very nice in ſuch Things, eſpecially amongſt Mechanics) than any Deſign of Impoſture.

By Mr. Tho. Luffkin. N. 255. *p.* 287. N. 266. *p.* 677. Aug. An. 1699.

2. Over-againſt the *Market-place* in *Colcheſter* ſtands the Houſe of Mr. *Furley*, a Linnen-draper; ſome of the backermoſt Part of which is an ancient *Roman* Building, but the Front is of leſſer ſtanding, and timbered. Upon the bottom Cell (which is almoſt in the Form of a triangular Priſm) of one of the Windows of the Front, between two carved Lions, ſtands an Eſcutcheon, containing only theſe Figures 1090. The Periphery of the Cyphers, and Nine, are rather fracted than flected, prominent, large, and very fair; but to make them the more perſpicuous, they are gilded by the Proprietor. The Window looks directly North, the Date being thereby preſerved from the ſcorching Heat of the Sun; and by its Inclination (falling from the *Vertex* or *Perpendicular* by an Angle of about 60 Degrees) from Rain, Snow, *&c.* If it be objected, that the ſecond and fourth Figures may repreſent that among the *Arabians*, which is with us as five; I anſwer, that the o is not uſed with all the *Arabs* for 5, but with ſome for a Cypher, and ſo it was uſed by the *Moors* in *Spain*, who firſt brought theſe Figures into our Parts; nor is the ſquare o an *Arabick* Letter, but an *Engliſh* Letter of that Age. And the Form of theſe Figures ſoon degenerated from that of the *Arabs*, into ſuch as we now uſe, if not at the firſt Reception from the *Arabs* [or *Moors*], certainly long before 1595, as this *Conſtruction* would make it. (*Fig.* 56.)

The Conſtruction of Logarithms; by Mr. Edm. Halley. N. 216. *p.* 58. Mar. An. 1695.

XXVIII. The old Definition of *Logarithms*, that they are the equally-differing Aſſociates of Proportional Numbers, is too ſcanty to define them fully:

For

For they may much more properly be ſaid to be *Numeri Rationum Exponentes*; wherein *Ratio* is conſidered as a *Quantitas ſui generis*, beginning from the Ratio of Equality, or 1 to 1 = 0; being Affirmative when the Ratio is increaſing, as of Unity to a greater Number, but Negative when decreaſing; and theſe *Rationes* we ſuppoſe to be meaſured by the Number of the *Ratiunculæ* contained in each. Now theſe Ratiunculæ are to be ſo underſtood as in a continued Scale of Proportionals, infinite in Number between the two Terms of the Ratio; which infinite Number of mean Proportionals is to that infinite Number of the like and equal Ratiunculæ between any other two Terms, as the Logarithm of the one Ratio is to the Logarithm of the other. Thus if there be ſuppoſed between 1 and 10 an infinite Scale of mean Proportionals, whoſe Number is 100000, *&c. in infinitum*; between 1 and 2 there ſhall be 30102, *&c.* of ſuch Proportionals, and between 1 and 3 there will be 47712, *&c.* of them; which Numbers therefore are the Logarithms of the Rationes of 1 to 10, 1 to 2, and 1 to 3; and not ſo properly to be called the Logarithms of 10, 2 and 3.

This being laid down, it is obvious that if between Unity and any Number propoſed, there be taken any Infinity of mean Proportionals, the infinitely little Augment or Decrement of the firſt of thoſe Means from Unity, will be a Ratiuncula, that is, the Momentum or Fluxion of the Ratio of Unity to the ſaid Number: And ſeeing that in theſe continual Proportionals all the Ratiunculæ are equal, their Sum, or the whole Ratio will be as the ſaid Momentum is directly; that is, the Logarithm of each Ratio will be as the Fluxion thereof. Wherefore if the Root of any infinite Power be extracted out of any Number, the Differentiola of the ſaid Root from Unity, ſhall be as the Logarithm of that Number. So that Logarithms thus produced, may be of as many Forms as you pleaſe to aſſume infinite Indices of the Power whoſe Root you ſeek: as if the Index be ſuppoſed 100000, &c. infinitely, the Roots ſhall be the Logarithms invented by the Lord *Napier*; but if the ſaid Index were 2302585, *&c.* Mr. *Briggs's* Logarithms would immediately be produced. And if you pleaſe to ſtop at any Number of Figures, and not to continue them on, it will ſuffice to aſſume an Index of a Figure or two more than your intended Logarithm is to have, as Mr. *Briggs* did, who to have his Logarithms true to 14 Places, by continual Extraction of the Square Root, at laſt came to have the Root of the 140737488355328th Power; but how operoſe that Extraction was, will be eaſily judged by whoſo ſhall undertake to examine his *Calculus*.

Now, tho' the Notion of an infinite Power may ſeem very ſtrange, and to thoſe that know the difficulty of the Extraction of the Roots of high Powers, perhaps impracticable; yet by the help of that admirable Invention of Mr. *Newton*, whereby he determines the Unciæ or Numbers prefix'd to the Members compoſing Powers, (on which chiefly depends the *Doctrine of Series*) the Infinity of the Index contributes to render the Expreſſion much more eaſy: For if the infinite Power to be reſolved be put (after Mr. *Newton*'s Method) $\overline{p+pq}$

 $\overline{p+pq}|^{\frac{1}{m}}$

$\overline{p+pq}|^{\frac{1}{m}}$, or $\overline{1+q}|^{\frac{1}{m}}$, instead of $1+\frac{1}{m}q+\frac{1-m}{2mm}qq+\frac{1-3m+2mm}{6m^3}q^3+\frac{1-6m+11mm-6m^3}{24m^4}q^4$, &c. (which is the Root when m is finite), becomes $1+\frac{1}{m}q-\frac{1}{2m}qq+\frac{1}{3m}q^3+\frac{1}{4m}q^4+\frac{1}{5m}q^5$, &c. mm being *infinitè* infinite; and consequently whatever is divided thereby vanishing. Hence it follows that $\frac{1}{m}$ multiplied into $q-\frac{1}{2}qq+\frac{1}{3}q^3-\frac{1}{4}q^4+\frac{1}{5}q^5$, &c. is the Augment of the first of our mean Proportionals between Unity and $1+q$, and is therefore the Logarithm of the Ratio of 1 to $1+q$; and whereas the Infinite Index m may be taken at pleasure, the several Scales of Logarithms to such Indices will be as $\frac{1}{m}$, or reciprocally as the Indices. And if the Index be taken 10000, *&c.* as in the case of *Napier*'s Logarithms, they will be simply $q-\frac{1}{2}qq+\frac{1}{3}q^3-\frac{1}{4}q^4+\frac{1}{5}q^5-\frac{1}{6}q^6$, &c.

Again; If the Logarithm of a decreasing Ratio be sought, the infinite Root of $1-q$, or $\overline{1-q}|^{\frac{1}{m}}$ is $1-\frac{1}{m}q-\frac{1}{2m}q^2-\frac{1}{3m}q^3-\frac{1}{4m}q^4-\frac{1}{5m}q^5-\frac{1}{6m}q^6$, &c. whence the Decrement of the first of our Infinite Number of Proportionals will be $\frac{1}{m}$ into $q+\frac{1}{2}qq+\frac{1}{3}q^3+\frac{1}{4}q^4+\frac{1}{5}q^5+\frac{1}{6}q^6$, &c. which therefore will be as the Logarithm of the Ratio of Unity to $1-q$: But if m be put 10000, *&c.* then the said Logarithm will be $q+\frac{1}{2}qq+\frac{1}{3}q^3+\frac{1}{4}q^4+\frac{1}{5}q^5+\frac{1}{6}q^6$, *&c.*

Hence the Terms of any Ratio being a and b, q becomes $\frac{b-a}{a}$ or the Difference divided by the lesser Term, when 'tis an increasing Ratio, or $\frac{b-a}{b}$ when 'tis decreasing, or as b to a; whence the Logarithm of the same Ratio may be doubly expressed; for, putting x for the Difference of the Terms a and b, it will be either

$\frac{1}{m}$

$$\frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2bb} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} \text{ \&c.}$$

$$\text{or } \frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2aa} + \frac{x^3}{3b^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} \text{ \&c.}$$

But if the Ratio of a to b be ſuppoſed to be divided into two Parts, *viz.* into the Ratio of a to the *Arithmetical Mean* between the Terms, and the Ratio of the ſaid *Arithmetical Mean* to the other Term b; then will the Sum of the Logarithms of thoſe two Rationes be the Logarithm of the Ratio of a to b; and, ſubſtituting $\frac{1}{2}z$ inſtead of $\frac{1}{2}a + \frac{1}{2}b$, the ſaid *Arithmetical Mean*, the Logarithms of thoſe Rationes will be, by the foregoing Rule,

$$\frac{1}{m} \text{ into } \frac{x}{z} + \frac{xx}{2zz} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} \text{ \&c.}$$

$$\text{and } \frac{1}{m} \text{ into } \frac{x}{z} - \frac{xx}{2zz} + \frac{z^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} \text{ \&c.}$$

the Sum whereof $\frac{1}{m}$ into $\frac{2x}{z} * + \frac{2x^3}{3z^3} * + \frac{2x^5}{5z^5} * + \frac{2x^7}{7z^7}$ &c. will be the Logarithm of the Ratio of a to b, whoſe Difference is x, and Sum z. And this Series converges twice as ſwift as the former, and therefore is more proper for the Practice of making Logarithms; which it performs with that Expedition, that where x the Difference is but the hundredth Part of the Sum, the firſt Step, $\frac{2x}{z}$, ſuffices to ſeven Places of the Logarithm, and the ſecond Step to twelve; but if *Briggs*'s firſt twenty *Chiliads* of Logarithms be ſuppoſed made, as he very carefully computed them, to fourteen Places, the firſt Step alone is capable to give the Logarithm of any intermediate Number, true to all the Places of thoſe Tables.

After the ſame Manner may the Difference of the ſaid two Logarithms be very aptly applied to find the Logarithm of prime Numbers, having the Logarithms of the two next Numbers above and below them: For the Difference of the Ratio of a to $\frac{1}{2}z$, and of $\frac{1}{2}z$ to b, is the Ratio of ab to $\frac{1}{4}zz$, and the Half of that Ratio is that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the Geometrical Mean to the Arithmetical. And conſequently the Logarithm thereof will be the Half-difference of the Logarithms of thoſe Rationes, *viz.*

$$\frac{1}{m} \text{ into } \frac{xx}{2zz} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x}{8z}, \text{ \&c.}$$

Which is a *Theorem* of good Diſpatch to find the Logarithm of $\frac{1}{2}z$. But the ſame is yet much more advantageouſly performed by a Rule derived from the foregoing, and beyond which, in my Opinion, nothing better can be hoped. For the Ratio of ab to $\frac{1}{4}zz$, or $\frac{1}{4}aa + \frac{1}{2}ab\frac{1}{2} + \frac{1}{4}bb$, has the Difference of its Terms, $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$, or the Square of $\frac{1}{2}a - \frac{1}{2}b = \frac{1}{4}xx$, which, in the preſent Caſe of finding the Logarithms of Prime Numbers,

3 is

is always Unity; and calling the Sum of the Terms $\frac{1}{4}zz + ab = yy$, the Logarithm of the Ratio of $\sqrt{ab}$ to $\frac{1}{2}a + \frac{1}{2}b$, or $\frac{1}{2}z$ will be found,

$$\frac{1}{m} \text{ into } \frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}}, \text{ \&c.}$$

which converges very much faſter than any *Theorem* hitherto publiſhed for this Purpoſe.

Here note, that $\frac{1}{m}$ is all along applied to adapt theſe Rules to all Sorts of Logarithms. If *m* be 10000, *&c.* it may be neglected, and you will have *Napier*'s Logarithms, as was hinted before; but if you deſire *Briggs*'s Logarithms, which are now generally received, you muſt divide your Series by 2, 302585092994045684017991454684364207601101488628772976033328, or multiply it by the Reciprocal thereof, *viz.* 0, 434294481903251827651128918916605082294397005803666566114454.

But to ſave ſo operoſe a Multiplication (which is more than all the reſt of the Work) it is expedient to divide this Multiplicator by the Powers of *z* or *y* continually, according to the Direction of the *Theorem*, eſpecially where *x* is ſmall and Integer, reſerving the proper Quotes to be added together, when you have produced your Logarithm to as many Figures as you deſire; of which Method I will give a Specimen, in the Logarithms of the firſt prime Numbers under 20 to ſixty Places, computed by Mr. *Abraham Sharp*, as they were communicated to me by our common Friend, Mr. *Euclid Speidal.*

Num.	Logarithms,
2.	0, 301029995663981195213738894724493026768189881462108541310427
3.	0, 477121254719662437295027903255115309200128864190695864829866
7.	0, 845098040014256830712216258592636193483572396323965406503835
11.	1, 041392685158225040750199971243024241706702190466453094596539
13.	1, 1139433523068377692065418950262462545611189000505367328859808 3
17.	1, 230448921378273028540169894328337030007567378425046397380368
19.	1, 278753600952828961536333475756929317951192337394497598906819

The next Prime Number is 23, which I will take for an Example of the foregoing Doctrine; and by the firſt Rules, the Logarithm of the Ratio of 22 to 23 will be found to be either

$$\frac{1}{22} - \frac{1}{968} + \frac{1}{31944} - \frac{1}{937024} + \frac{1}{25768160}, \text{ \&c.}$$

$$\text{or } \frac{1}{23} + \frac{1}{1058} + \frac{1}{36531} + \frac{1}{1119364} + \frac{1}{32181715}, \text{ \&c.}$$

As

As likewise that of the Ratio of 23 to 24 by a like Process.

$$\frac{1}{23}-\frac{1}{1058}+\frac{1}{36501}-\frac{1}{1119364}+\frac{1}{32181715}, \text{ \&c. or}$$

$$\frac{1}{24}+\frac{1}{1152}+\frac{1}{41471}+\frac{1}{1327104}+\frac{1}{39813120}, \text{ \&c.}$$

And this is the Result of the Doctrine of *Mercator*, as improved by the learned Dr. *Wallis*. But by the second *Theorem*, viz. $\frac{2x}{z}+\frac{2x^3}{3z^3}+\frac{2x^5}{5z^5}$, *&c.* the same Logarithms are obtained by fewer Steps; to wit,

$$\frac{2}{45}+\frac{2}{273375}+\frac{2}{922640625}+\frac{2}{2615686171875}, \text{ \&c. And}$$

$$\frac{2}{47}+\frac{2}{311469}+\frac{2}{1146725035}+\frac{2}{3546361843241}, \text{ \&c.}$$

which was invented and demonstrated in the Hyperbolical Spaces analogous to the Logarithms, by the excellent Mr. *James Gregory*, in his *Exercitationes Geometricæ*, and since further prosecuted by the aforesaid Mr. *Speidall*, in a late Treatise in *English* by him published on this Subject. But the Demonstration, as I conceive, was never till now perfected without the Consideration of the Hyperbola, which in a Matter purely Arithmetical, as this is, cannot so properly be applied. But what follows, I think, I may more justly claim as my own, *viz. That the Logarithm of the Ratio of the Geometrical Mean to the Arithmetical*, between 22 and 24, or of $\surd$ 528 to 23, will be found to be either,

$$\frac{1}{1058}+\frac{1}{1119364}+\frac{1}{888215334}+\frac{1}{626487882248}, \text{ \&c. or}$$

$$\frac{1}{1057}+\frac{1}{3542796579}+\frac{1}{6596765584852\,85}, \text{ \&c.}$$

All these Series being to be multiplied into 0,4342944819, *&c.* if you design to make the *Logarithm* of *Briggs*. But with great Advantage in respect of the Work, the said 0,4342944819, *&c.* is divided by 1057, and the Quotient thereof again divided by three times the Square of 1057, and that Quotient again by $\frac{3}{5}$ of that Square, and that Quotient by $\frac{5}{7}$ thereof, and so forth, till you have as many Figures of your Logarithm as you desire. As for Example, the Logarithm of the Geometrical Mean between 22 and 24 is found by the Logarithms of 2, 3, and 11, to be

		1,3613169612669061294500917266980$_5$
1057)	43429 &c.	(- - - 41087462810146814347315886368
3 *in* 1117249)	41087 &c.	(- - - - - - - 1225852154418182946007$_4$
$\frac{5}{?}$ *in* 1117249)	12258 &c.	(- - - - - - - - - - - - - 6583235184376175
$\frac{?}{5}$ *in* 1117249)	65832 &c.	(- - - - - - - - - - - - - - - - - - - 4208829765
$\frac{6}{7}$ *in* 1117249)	42088 &c.	(- 2930
	Summa	1,3617278360175728788677711225119

Which is the Logarithm of 23 to thirty-two Places, and obtained by five Divisions with very ſmall Diviſors; all which is much leſs Work than ſimply multiplying the Series into the ſaid Multiplicator 0,43429, &c.

From the Logarithm given to find what Ratio it expreſſes, is a *Problem* that has not been ſo much conſidered as the former, but which is ſolved with the like Eaſe, and demonſtrated by a like Proceſs, from the ſame general *Theorem* of Mr. *Newton*; for as the Logarithm of the Ratio of 1 to $1 + q$ was proved to be $\overline{1+q}|^{\frac{1}{m}} - 1$, and that of the Ratio of 1 to $1 - q$ to be $1 - \overline{1-q}|^{\frac{1}{m}}$: ſo the Logarithm, which we will from henceforth call L, being given, $1 + L$ will be equal to $\overline{1+q}|^{\frac{1}{m}}$ in the one Caſe, and $1 - L$ will be equal to $\overline{1-q}|^{\frac{1}{m}}$ in the other: Conſequently $\overline{1+L}|^{m}$ will be equal to $1 + q$, and $\overline{1-L}|^{m}$ to $1 - q$; that is, according to Mr. *Newton*'s ſaid Rule, $1 + mL + \frac{1}{2}m^2L^2 + \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4 + \frac{1}{120}m^5L^5$, &c. will be = to $1 + q$; and $1 - mL + \frac{1}{2}m^2L^2 - \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4 - \frac{1}{120}m^5L^5$, &c. will be equal to $1 - q$; m being any infinite Index whatſoever: which is a full and general *Propoſition* from the Logarithm given to find the Number, be the Species of Logarithms what it will. But if *Napier*'s Logarithm be given, the Multiplication by m is ſaved, (which Multiplication is indeed no other than the reducing the other Species to his) and the Series will be more ſimple, *viz.* $1 + L + \frac{1}{2}L^2 + \frac{1}{6}L^3 + \frac{1}{24}L^4 + \frac{1}{120}L^5$, &c. or $1 - L + \frac{1}{2}L^2 - \frac{1}{6}L^3 + \frac{1}{24}L^4 - \frac{1}{120}L^5$, &c. This Series, eſpecially in great Numbers, converges ſo ſlowly, that it were to be wiſhed it could be contracted.

If one Term of the Ratio, whereof L is the Logarithm, be given, the other Term will be had eaſily by the ſame Rule: For if L were *Napier*'s Logarithm

rithm of the Ratio of a the leſſer to b the greater Term, b would be the Product of a into $1 + L + \frac{1}{2} L^2 + \frac{1}{6} L^3$, &c. $= a + aL + \frac{1}{2} aL^2 + \frac{1}{6} aL^3$, &c. But if b were given, a would be $= b - bL + \frac{1}{2} bL^2 - \frac{1}{6} bL^3$, &c. Whence, by the help of the *Chiliads*, the Number appertaining to any Logarithm will be exactly had to the utmoſt Extent of the Tables. If you ſeek the neareſt next Logarithm, whether greater or leſſer, and call its Number a if leſſer, or b if greater; then the given L, and the Difference thereof from the ſaid neareſt Logarithm you call l; it will follow that the Number anſwering to the Logarithm L will be either a into $1 + l + \frac{1}{2} l^2 + \frac{1}{6} l^3 + \frac{1}{24} l^4 + \frac{1}{120} l^5$, &c. or elſe b into $1 - l + \frac{1}{2} l^2 - \frac{1}{6} l^3 + \frac{1}{24} l^4 - \frac{1}{120} l^5$, &c. wherein as l is leſs, the Series will converge the ſwifter. And if the firſt 20000 Logarithms be given to 14 Places, there is rarely occaſion for the three firſt Steps of this Series to find the Number to as many Places. But for *Ulacq*'s great *Canon* of 100000 Logarithms, which is made but to ten Places, there is ſcarce ever need for more than the firſt Step $a + al$, or $a + mal$ in one Caſe, or elſe $b - bl$, or $b - mbl$ in the other, to have the Number true to as many Figures as thoſe Logarithms conſiſt of.

There is another Series which is not indeed ſo ſimple and uniform, yet the firſt Step thereof is moſt commodious for Practice, and exact enough for Tables not exceeding 14 Places: It is thus; $a + \frac{al}{1 - \frac{1}{2}l}$ or $b - \frac{bl}{1 + \frac{1}{2}l}$ will be the Number anſwering to the Logarithm given, differing from the Truth but by one half of the third Step from the former Series. But that which renders it yet more eligible is, that with equal Facility it ſerves for *Briggs*'s or any other ſort of Logarithms, with the only Variation of writing $\frac{1}{m}$ inſtead of 1, that is

$$a + \frac{al}{\frac{1}{m} - \frac{1}{2}l}, \text{ and } b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l} \text{ or } \frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l} \text{ and } \frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l},$$

which are eaſily reſolved into Analogies, *viz.*

As 42429 &c. $- \frac{1}{2}l$: to 43429 $+ \frac{1}{2}l$: : ſo is a :
Or, As 43429 &c. $+ \frac{1}{2}l$: to 43429 $- \frac{1}{2}l$: : ſo is b :
} to the Number ſought.

If more of this Series be desired, it will be found as follows,

$$a + \frac{a\,l}{2 - \frac{1}{2}l} - \frac{\frac{1}{12}a\,l^3}{1 - l} + \frac{\frac{1}{24}a\,l^5}{1 - 2\,l},$$ *&c.* as may easily be demonstrated by working out the Divisions in each Step, and collecting the Quotes, whose Sum will be found to agree with our former Series; which is no other than an easy *Corollary* to Mr. *Newton*'s general *Theorem* for forming *Roots* and *Powers*.

XXIX. *Papers of less General Use omitted.*

Tangents to Curves. N. 81. p. 4010. Mar. An. 1672.

1. A Breviat of Dr. *Wallis*'s two Methods of drawing Tangents; Extracted by him from his *Con. Sect.* and other Parts of his *Mathematical Works*.

Rectification of Curves. N. 98. p. 6146, 6149. Nov. An. 1673.

2. Mr. *Huygens* in his *Hor. Oscill.* having given M. *Huraet* the Honour of inventing a Curve equal to a Straight Line in the Year 1659; Dr. *Wallis* here asserts this Invention to Mr. *William Neile* (Son of Sir *Paul Neile*), who discovered and demonstrated the Equality of a *Paraboloid* to a Straight Line two Years before. The same was soon after otherwise demonstrated by my Lord *Brounker*, and Sir *Christopher Wren*, in *June* and *July* 1657; and the Demonstrations inserted by Dr. *Wallis*, in his Tract *de Cycloide* 1659, with a fair Relation of the whole Matter. Besides, Sir *Christopher Wren* found a straight Line equal to that of a *Cycloid* in the Year 1658: Yet he freely confesses Mr. *Neile*'s Invention of a Curve capable of *Rectification* the Year before.

Ib. p. 6150.

Transformation of Curves. N. 214. p. 233. Nov. An. 1694.

3. The Abbot *Galloys*, having, in the Year 1693, asserted that Mr. *James Gregory* and Dr. *Barrow* stole their General Propositions concerning the *Transformation of Curves* from Mr. *Robervall*; Dr. *David Gregory* here fully refutes that Assertion. For Mr. *Gregory* published his Book at *Padua* 1668, and Dr. *Barrow* his *Lectiones Geometricæ* 1674, which Mr. *Robervall* doubtless had a Sight of before he died (which was not till *October* 1675), yet he never complained of any such Injury done him.

Cycloidal Spaces perfectly Quadrable. N. 217. p. 111. Oct. An. 1695.

4. Besides that Segment of the *Semicycloidal Figure*, first observed by Sir *Christopher Wren*, and after him by Mr. *Huygens*, and a Trilinear Part of it, which are capable of being Geometrically Squared; Dr. *Wallis* here produces from his Tracts *de Cycloide*, and *de Motu*, some other Portions thereof equally capable of Quadrature.

The Cycloid consider'd long ago. N. 229. p. 561. June An. 1697.

5. Dr. *Wallis* finds among the Mathematical Works of *Bovillus*, published at several times between the Years 1501 and 1510, that the Curve (which is now called the *Cycloid*,) was then considered. But he also finds that *Bovillus* was not the first who considered it: For Cardinal *Cusanus*, as appears by an ancient Manuscript of his Works (transcribed by *J. Scoblant* in the Year 1451) had considered it some time before. The Figure indeed (thro' the Unskilfulness of the Transcriber) both in the MS. and the *Basil* Edition, A. 1565, is very ill drawn; but being corrected according to the true Meaning of that *Cardinal*'s own Words, it evidently represents the modern *Cycloid*. From hence 'tis manifest, that this Curve was not first taken into Consideration either

either by *Merſennus* or *Galilæo*, but ſome Ages before, tho' never well underſtood till this preſent Age.

6. Some Papers ſent by Mr. *Jo. Collins* to Dr. *Wallis*, giving his Thoughts about ſome Defects in Algebra; which he did not live to finiſh. *Defects in Algebra.* N. 159. *p.* 575. May An. 1684.

XXX. *Accounts of Books, with Editions, Emendations,* &c. *omitted.*

1. EUclidis *Elementa Geometrica, novo ordine ac methodo demonſtrata.* Lond. 1666. N. 15. *p.* 261.

2. Archimedis *Opera*; Apollonii Perg: *Conic. Libri* iv; Theodoſii *Sphærica, Methodo nova illuſtrata, & ſuccincte demonſtrata: ab* Iſa. Barrow, *R. S. S.* Lond. 1675. *in 4to.* N. 114. *p.* 314.

3. Ἀρχιμήδȣς τȣ̃ Συρακȣσίȣ Ψαμμίτης, κ̀ Κύκλȣ μέτρησις: Εὐτοκίȣ Ἀσκαλωνίτȣ εἰς αὐτὴν ὑπόμνημα, &c. *Cum Verſione & Notis* Jo. Wallis, S. S. Th. D. *Oxon.* 1676. N. 123. *p.* 567.

4. *Theon Smyrnæus*, publiſhed at *Paris* by *Iſmael Bulialdus* in Greek and Latin. N. 80. *p.* 3095.

5. Diophanti Alexandrini *Arithmeticorum Libri ſex, & de Numeris Multangulis Liber unus*; *cum Commentariis* C. G. Bacheti, *& Obſervationibus* D. P. de Fermat, *Senatoris* Tholoſani: *cui acceſſit Doctrinæ Analyticæ Inventum Novum.* Toloſæ 1670. *in Folio.* N. 72. *p.* 2185.

6. The Works of Monſieur *de Fermat.* N. 1. *p.* 15.

7. Franciſci du Laurens *Specimina Mathematica, duobus Libris comprehenſa. Horum prior Syntheticus agit de Genuinis Matheſeos Principiis in genere*; *in ſpecie autem de veris Geometriæ Elementis hucuſque nondum traditis. Poſterior Analyticus de Methodo Compoſitionis atque Reſolutionis fuſe differit, & multa nova complectitur, quæ ſubtiliſſimam Analyſeos Artem mirum in modum promovent.* This Book is here cenſured, ſome Miſtakes in it corrected, and the Cenſure vindicated, by Dr. *Wallis.* N. 30. 580. N. 34. p. 654. N. 38. *p.* 744. N. 39. *p.* 775. N. 41. *p.* 825.

8. R. P. Andreæ Taquet, *è S. J. Opera Mathematica.* Antwerp. 1669. in *Folio.* N. 43. *p.* 869.

9. A Mathematical Compendium, collected out of the Notes and Papers of Sir *Jonas Moore*, by *Nicholas Stevenſon. Lond.* 1674. in 12*mo.* N. 104. *p.* 83.

10. R. P. Claudii Franc. Milliet de Chales, è S. J. *Curſus ſeu Mundus Mathematicus, univerſam Matheſin tribus Tomis complectens.* Lugd. 1674. in *Folio.* N. 110. *p.* 229.

11. The Mathematical Works of Dr. *Jo. Wallis, Savilian Profeſſor of Geometry* in the Univerſity of *Oxford*, & F. R. S. in three Volumes in *Folio. Oxon.* N. 216. *p.* 73. N. 154. *p.* 259.

12. An Introduction to Algebra, tranſlated out of *High Dutch* into *Engliſh* by *Tho. Branker*, M. A. much altered and augmented by Dr. *J. Pell.* Alſo a Table of ſuch odd Numbers as are leſs than One Hundred Thouſand, ſhewing thoſe that are *Incompoſit*, and reſolving the reſt into their *Factors*, or *Coefficients. Lond.* in 4*to.* N. 35. *p.* 688.

13. *Labyrinthus Algebræ. Auth.* Joh. Jac. Ferguſon. 1667. in 4*to.* N. 49. *p.* 996.

N. 90. p. 5152. 14. The Elements of that Mathematical Science called *Algebra*; by *Jo.*
N. 95. p. 6073. N. 108. p. 192. *Kerſey*. Lond. 1673. in *Folio.*
N. 143. p. 21. 15. A Treatiſe of *Algebra*, both Hiſtorical and Practical; by *Jo. Wallis*,
N. 173. p. 1095. N. 233. p. 730. D. D. In the 109th Chapter there are ſome Numbers miſtaken, which are here rectified by the Author.

N. 14. p. 253. 16. *De Principiis & Ratiocinatione Geometrarum; contra Faſtum Profeſſo-*
N. 16. p. 289. *rum Geometriæ. Authore* Tho. Hobbes. This Book is here animadverted on, and anſwered, by Dr. *Wallis.*

N. 48. p. 971. 17. Thomæ Hobbes *Quadratura Circuli, Cubatio Sphæræ, Duplicatio Cubi, confutata. Auth.* Jo. Wallis. S. T. D. Oxon. 1669. in *Quarto.*

N. 55. p. 1121. 18. Thomæ Hobbes *Quadratura Circuli, Cubatio Sphæræ, Duplicatio Cubi (ſecundo edita) denuo refutata. Auth.* Jo. Wallis. S. T. D. Oxon. 1669.

N. 72. p. 2185. 19. *Roſetum Geometricum, cum Cenſura brevi* Doctrinæ Walliſianæ *de Motu. Auth.* Tho. Hobbes *Malmeſburienſi*. Lond. 1671. in *Quarto.* This Book is
N. 73. p. 2202. here anſwered by Dr. *Wallis.*

N. 75. p. 2241. 20. Four Papers of Mr. *Hobbs*'s, publiſhed in the Months of *Auguſt* and *September* 1671. which are here anſwered by —— —

N. 86. p. 5047. 21. *Lux Mathematica, Colliſionibus* Johannis Walliſii, S. T. D. & Thomæ Hobbeſii Malmſburienſis, *excuſſa, multis & fulgentiſſimis aucta radiis. Auth.* R. R. *Adjuncta Cenſura Doctrinæ* Walliſianæ *de Libra, una cum* Roſeto Hobbe-
N. 87. p. 5067. ſii. Lond. 1672. in *Quarto.* This Book is here anſwered by Dr. *Wallis.*

N. 97. p. 6131. 22. *Principia & Problemata aliquot Geometrica, ante deſperata, nunc breviter explicata & demonſtrata. Auth.* T. H. Malmsburienſi. Lond. 1673. in *Quarto.*

N. 185. p. 245. 23. *Le Grand & Fameux Probleme de la Quadrature du Cercle reſolu Geometriquement par le Cercle & la Ligne droit, par* M. Mallement de Meſſange. *à* Paris. 1686. in *Twelves.* This Book is here refuted by M. D. *Cluverius.* R. S. S.

N. 32. p. 625. 24. *Nouveaux Elemens de Geometrie:* Or a Mathematical Treatiſe, entituled *New Elements of Geometry.* Paris 1667. in *Quarto.*

N. 79. p. 3064. 25. *Elemens de Geometrie*; *par le* P. Ignace Gaſton Pardies, *de la Comp. de J.* à Paris 1671. in *Twelves.*

N. 33. p. 640. 26. 1. *Vera Circuli & Hyperbolæ Quadratura, in propria ſua Proportionis Specie inventa & Demonſtrata*, à Jac. Gregorio Scoto. Patavii. in *Quarto.*
Ib. p. 641. This Subject is here further conſidered, and the Area of an Hyperbole explained; by Mr. *J. Collins.*

N. 37. p. 732. 2. M. *Huygens* having publiſhed Animadverſions upon this Book, in the *Journal de Sçavans*, 1668. Mr. *Gregory* here anſwers them. To this M. *Huy-*
N. 44. p. 882. *gens* replied in a following Journal of that Year; and Mr. *Gregory*, further to elucidate the Controverſy, here returns a ſecond Anſwer.

N. 216. p. 65. 3. In the 48th Page of this Book, Mr. *Halley* has diſcovered and corrected a ſmall Miſtake in the Logarithm of 10.

N. 35. p. 685. 27. *Geometriæ pars Univerſalis, Quantitatum Curvarum Tranſmutationi & Menſuræ inſerviens. Auth.* Jac. Gregorio *Scoto.* Patavii 1668. in *Quarto.*

N. 37. p. 378. 28. *De Infinitis Spiralibus inverſis, Infinitiſque Hyperbolis, aliiſque Geometricis. Auth.* F. Stephano de Angelis *Veneto.* Patavii. in *Quarto.*

29. *Michaelis Angeli Ricci Exercitatio Geometrica.* Romæ. in 4*to*. Reprinted at *London*, and annexed to *Mercator*'s *Logarithmotechnia.* N. 37. p. 738.

30. Renati Franc. Slufii *Mefolabum. Cui acceffit pars altera de Analyfi, & Mifcellanea.* Leodii Eburonum 1668. in 4*to*. N. 45. p. 903.

31. *Elementa Geometriæ Planæ. Authore* Ægidio Francifco de Gottignies *Bruxellenfi. S. J.* Romæ. 1669. in 12*mo*. N. 67. p. 2054.

32. *Synopfis Geometrica; cum tribus Opufculis, de Linea Sinuum & Cycloide; de Maximis & Minimis, Centuria; & Synopfis Geometriæ Planæ. Auth.* Honor. Fabry. *S. J.* Lugduni Galliarum 1669. in 12*mo*. N. 67. p. 2054.

33. *Lectiones* xiii. *Geometricæ; in quibus (præfertim) Generalia Linearum Curvarum Symptomata declarantur, ab* Ifaaco Barrow. Lond. 1669. in 4*to*. To thefe Lectures the Author here adds feveral *Corollaries* and *Theorems*. N. 75. p. 2260.

34. Erafmi Bartholini *Selecta Geometrica.* Hauniæ. 1674. in 4*to*. N. 106. p. 137.

35. *Elemens des Mathematiques, ou Principes Géneraux de toutes les Sciences qui ont les Grandeurs pour Object. Par* J. P. à Paris 1675. in 4*to*. N. 126. p. 638.

36. *Nouvelle Methode en Geometrie pour les Sections des Superficies Coniques & Cylindriques; qui ont pour Bafe des Circles, ou des Paraboles, des Ellipfes, & des Hyperboliques; par* Ph. de la Hire. à Paris 1673. in 4*to*. N. 129. p. 745.

37. *De Cycloide & Sectionibus Conicis.* Ph. de la Hire. Ibid. p. 746.

38. The Geometrical Key, or Conftruction of all Equations, Linear, Quadratic, Cubic, and Biquadratic, by a Circle and one only Parabola; by Mr. *Tho. Baker.* N. 157. p. 549.

39. *Exercitatio Geometrica de Dimenfione Figurarum. Auth.* Davide Gregorio. Edinb. 1684. in 4*to*. N. 163. p. 730.

40. *Methodus Figurarum Lineis Rectis & Curvis comprehenfarum Quadraturas determinandi, Auth.* J. Craig. Lond. 1685. in 4*to*. To this Tract the Author here makes an Addition; and takes notice of fome Remarks made on it in the *Act. Lipf.* by M. *Leibnitz*, and M. *J. Bernoulli.* N. 183. p. 185. Ibid. p. 186. N. 235. p. 786.

41. *Tractatus Mathematicus de Figurarum Curvilinearum Quadraturis & Locis Geometricis. Auth.* J. Craig. Lond. 1693. in 4*to*. N. 209. p. 113.

42. *Tractatus de Principiis Calculi Exponentialis. Auth.* D. Bernoullio; wherein a Miftake is here difcovered and corrected, by Mr. *Craig.* N. 245. p. 374.

43. *Analyfis Geometrica, five nova & vera Methodus Refolvendi, tam Problemata Geometrica, quam Arithmeticas Quæftiones. Pars prima, de Planis. Auth.* D. Antonio Hugone de Omerique *Sanlucarenfe.* N. 257. p. 351.

44. Stereometrical Propofitions, varioufly applicable, but particularly intended for *Gauging*; by *Rob. Anderfon. Lond.* 1668. in 8*vo*. N. 39. p. 785.

45. *Gauging* promoted; being an Appendix to Stereometrical Propofitions; by *Rob. Anderfon. Lond.* 1669. in 8*vo*. N. 47. p. 960.

46. *Gauging* Epitomized; by *Mich. Dary. Lond.* 1699. upon one Folio Page. N. 52. p. 1054.

47. *Tabula Numerorum Quadratorum decies Millium, una cum ipforum Lateribus ab Unitate incipientibus, & Ordine Naturali ufque ad* 10000 *progredientibus.* Lond. 1672. N. 82. p. 4050.

48. The Defcription and Ufe of two Arithmetick Inftruments, &c. by Sir *Sam. Moreland. Lond.* 1673. N. 94. p. 6048.

49. Johannis

N. 139. p. 980. 49. Johannis Wallisii S. T. D. *Exercitationes tres: 1. De Cometarum Distantiis investigandis. 2. De Rationum & Fractionum Reductione. 3. De Periodo Juliano.* Lond. 1678.

N. 38. p. 753. Ibid. p. 756. Ibid. p. 759. 50. *Logarithmotechnia* Nicholai Mercatoris. Lond. 1668. in *Quarto*. This Author's Method of squaring the Hyperbola, and of finding the Sum of the Logarithms is here improved by Dr. *Wallis*; and further explicated by the Author himself.

CHAP. II.

Trigonometry, Surveying.

A Chronographical Problem proposed by Mr. Rich. Townley, *solved by Mr.* John Collins. N. 69. p. 2093. Mar. An. 1674.

I. Prob. *THE Distances of three Objects in the same Plane being given, as A, B, C; the Angles made at a fourth Place in the same Plane, as at S, are observed: the Distances from the Place of Observation to the respective Objects are required.*

The *Problem* hath six *Cases*.

Fig. 57. *CASE* 1. If the Station be taken without the Triangle made by the Objects, but in one of the Sides thereof produced, as at S: find the Angle A C B; then in the Triangle A C S, all the Angles and the Side A C are known; whence either or both the Distances S A, or S C, may be found.

Fig. 58. *Case* 2. If the Station be in one of the Sides of the Triangle, as at S: then having the three Sides, A C, C B, B A, given, find the Angle C A B; then again in the Triangle S A B all the Angles, and the Side A B are known; whence may be found either A S, or S B, Geometrically; if you make the Angle C A D equal to the observed Angle C S B, and draw B S parallel to D A, you determine the Point of Station S.

Fig. 59. *Case* 3. If the three Objects lie in a right Line as A C D (suppose it done), and that a Circle passeth through the Station S, and the two Exterior Objects A, B: then is the Angle A B D equal to the observed Angle A S C (by 21. 3. *E.*) as insisting on the same Arch A D; and the Angle B A D in like manner equal to the observed Angle C S B: By this means the Point D is determined. Join D C, and produce the same, then a Circle passing through the Points A, B, D, intersects D C produced, at S, the Place of Station.

Calculation.] In the Triangle A B D, all the Angles and the Side A B, are known, whence may be found the Side A D.

Then in the Triangle C A D, the two Sides C A, and A D, are known, and their contained Angle C A D is known; whence may be found the Angles C D A, and A C D, the Complement whereof to a Semicircle is the Angle S C A: in which Triangle the Angles are now all known, and the Side A C: whence may be found either of the Distances S C, or S A.

Case

Case 4. If the Station be without the Triangle made by the Objects, the Sum of the Angles observed is less than four Right Angles. The *Construction* is the same as in the last *Case*, and the *Calculation* likewise; saving that you must make one Operation more; having the three Sides A C, C B, B A, thereby find the Angle C A B, which add to the Angle E A D, then you have the two Sides, *viz.* A C, being one of the Distances, and A D, (found as in the former *Case*) with their contained Angle C A D, given, to find the Angles C D A, and A C D, the Complement whereof to a Semicircle is the Angle S C A: Now in the Triangle S C A, the Angle at C being found, and at S observed, and given by Supposition, the other at A is likewise known, as being the Complement of the two former to a Semicircle, and the Side A C given; hence the Distances C S, or A S, may be found. *Fig.* 60.

Case 5. If the Place of Station be at some Point within the Plane of the Triangle, made by the three Objects, the *Construction* and *Calculation* are the same as in the last, saving only that instead of the observed Angle A S C, the Angle A B D is equal to the Complement thereof to a Semicircle, to wit, it is equal to the Angle A S D; both of them insisting on the same Arch A D: And in like manner the Angle B A D is equal to the Angle D S B, which is the Complement of the observed C S B; and in this *Case* the Sum of the three Angles observed, is equal to four Right Angles. *Fig.* 61.

In these three latter *Cases* no use is made of the Angle observed between the two Objects, as A and B, that are made the Base-line of the *Construction*; yet the same is of ready use for finding the third Distance or last Side sought; as in the Triangle S A B, there is given the Distance A B, its opposite Angle equal to the Sum of the two observed Angles, and the Angle S A B attained, as in the fourth *Case*: Hence the third Side, or last Distance S B, may be found. *Fig.* 60.

And here it may be noted, that the three Angles C A S, A S B, S B C, are together equal to the Angle A C B; for the two Angles C S B and C B S are equal to E C B, as being the Complement of S C B to two Right Angles; and the like in the Triangle on the other Side. *Ergo*, &c.

Case 6. If the three Objects be A, B, C, and the Station at S, as before, it may happen, according to the former *Constructions*, that the Points C and D may fall close together, and so a right Line joining them, shall be produced with Uncertainty; in such *Case* the Circle may be conceived to pass through the Place of Station at S, and any two of the Objects, as through B and C; wherein making the Angle D B C equal to the observed Angle A S C, and B C D equal to the Complement to 180 deg. of both the observed Angles in D S B; thereby the Point D is determined, through which, and the Points C, B, the Circle is to be described; and joining D A, (produced when Need requireth) where it intersects the Circle, as at S, is the Place of Station sought. *Fig.* 62.

This *Problem* may be of good use for the due Situation of Sands and Rocks, that are within sight of three Places upon Land, whose Distances are well known; or for *Chorographical Uses*, &c. especially now there is a Method of observing Angles nicely accurate by the Aid of a Telescope.

II. The

Three Chorographic Problems solved by a Member of the Philosophical Society at Oxford. N. 177. p. 1231. *Dec.* An. 1685.

II. The three following *Problems* may occur at Sea, in finding the Distance and Position of Rocks, Sands, *&c.* from the Shore; or in Surveying the Sea-Coast; when only two Objects, whose Distance from each other is known, can be seen at one Station: But especially they may be useful to one, that would make a Map of a Country by a Series of Triangles derived from one or more measured Bases; which is the most exact Way of finding the Bearing and Distance of Places from each other, and thence their true Longitude and Latitude; and may consequently occur to one that would in that Matter measure a Degree on the Earth.

Fig. 63.

Prob. 1.] *There are two Objects B and C, whose Distance B C is known; and there are two Stations at A and E, where the Objects B, C, being visible, and the Stations one from another, the Angles B A C, B A E, A E B, A E C, are known by Observation,* (which may be made with an ordinary *Surveying Semicircle,* or *Cross-Staff*; or, if the Objects are beyond the View of the naked Eye, with a *Telescopic Quadrant*;) *To find the Distances or Lines A B, A C, A E, E C.*

Fig. 64.

Construction.] In each of the Triangles, B A E, C A E, two Angles at A, E, being known, the third is also known; then take any Line $\alpha\varepsilon$ at pleasure, on which constitute the Triangles, $\beta\alpha\varepsilon$, $\alpha\varepsilon\gamma$, respectively equiangular to the Triangles B A E, A E C; join $\beta\gamma$: Then upon B C constitute the Triangles B C A, B C E, equiangular to the correspondent Triangles $\beta\gamma\alpha$, $\beta\gamma\varepsilon$, join A E, and the Thing is manifestly done.

The Calculation.] Assuming $\alpha\varepsilon$, of any Number of Parts, in the Triangles $\alpha\beta\varepsilon$, $\alpha\gamma\varepsilon$, the Angles being given, the Sides $\alpha\beta$, $\alpha\gamma$, $\varepsilon\beta$, $\varepsilon\gamma$, may be found by Trigonometry: Then in the Triangle $\beta\alpha\gamma$, having the Angle $\beta\alpha\gamma$, and the Legs $\alpha\beta$, $\alpha\gamma$, we may find $\beta\gamma$. Then $\beta\gamma : BC :: \beta\alpha : BA :: \beta\varepsilon : BE :: \gamma\alpha : CA :: \gamma\varepsilon : CE$.

Fig. 65.

Prob. 2.] *Three Objects B, C, D, are given, or (which is the same) the Sides and consequent Angles of the Triangle B C D are given; also there are two Points or Stations A, E, such that at A may be seen the three Points B, C, E, but not D, and at the Station E, may be seen A, C, D, but not B; that is, the Angles B A C, B A E, A E C, A E D, (and consequently E A C, A E C) are known by Observation: To find the Lines A B, A C, A E, E C, E D.*

Fig. 66.

Construction.] Take any Line $\alpha\varepsilon$ at pleasure, and at its Extremities make the Angles $\varepsilon\alpha\gamma$, $\varepsilon\alpha\beta$, $\alpha\varepsilon\gamma$, $\alpha\varepsilon\delta$, equal to the correspondent observed Angles E A C, E A B, A E C, A E D. Produce $\beta\alpha$, $\delta\varepsilon$, till they meet in ϕ; join $\phi\gamma$: then upon C B describe (according to 33. 3. *E.*) a Segment of a Circle, that may contain an Angle $= \gamma\phi\beta$; and upon C D describe a Segment of a Circle capable of an Angle $= \gamma\phi\delta$: Suppose F the common Section of these 2 Circles; join F B, F C, F D; then from the Point C, draw for the Lines C A, C E, so that the Angle F C A may be $= \phi\gamma\alpha$, and F C E $= \phi\gamma\varepsilon$: so A, E, the common Sections of C A, C E, with F B, F D, will be the Points required, from whence the rest is easily deduced.

Calculation.]

Calculation.] Assuming $\alpha\epsilon$ of any Number, in the Triangles $\alpha\gamma$, $\alpha\epsilon$, $\phi\epsilon$, all the Angles being given, with the Side $\alpha\epsilon$ assumed, the Sides $\alpha\gamma$, $\epsilon\gamma$, $\alpha\phi$, $\epsilon\phi$, will be known; then in the Triangle $\gamma\alpha\phi$, the Angle $\gamma\alpha\phi$, with the Legs $\alpha\gamma$, $\alpha\phi$, being known, the Angles $\alpha\phi\gamma$, $\alpha\gamma\phi$, with the Side $\phi\gamma$, will be known: Then as for the rest of the Work, the Triangle BCD having all its Sides and Angles known, and the Angles BFC, BFD, being equal to the found $\beta\phi\gamma$, $\beta\phi\delta$; how to find FB, FC, FD, by *Calculation* (and also *Protraction*) has been already shewn above by Mr. *Collins*, as to all its *Cases*. *Sect.* 1.

But it must here be noted, that if the Sum of the observed Angles BAE, AED, is 180 deg. then AB, and ED, cannot meet, because they are parallel, and consequently the given Solution cannot take Place; for which Reason I here subjoin another.

Another Solution.] Upon BC describe a Segment BAC, of a Circle, so that the Angle of the Segment may be equal to the observed Angle $\beta\alpha\gamma$, (which is shewn 33. 3. *E.*) and upon CD describe a Segment CED, of a Circle, capable of an Angle equal to the observed CED; from C draw the Diameters of these Circles CG, CH; then upon CG describe a Segment of a Circle GFC, capable of an Angle equal to the observed Angle ABC; likewise upon CH, describe a Circle's Segment CFA, capable of an Angle equal to the observed Angle CAE: suppose F the common Section of the two last Circles HFC, GFC; join FH, cutting the Circle HEC in E; join also FG, cutting the Circle GAC in A: I say, that A, E, are the Points required. *Fig.* 67.

Demonstration.] For the Angle BAC is $= \beta\alpha\gamma$, by *Construction* of the Segment; also the Angles CEH, CAG, are right, because each exists in a Semicircle: Therefore a Circle being described upon CF, as a Diameter, will pass thro' E, A; therefore the Angle CAE $=$ CFE $=$ CEH $=$ (by *Construction*) to the observed Angle $\gamma\alpha\epsilon$. In like manner the Angle CEA $=$ CFA $=$ CFG $=$ observ'd Angle $\gamma\epsilon\alpha$.

If the Stations A, E, fall in a right Line with the Point C; the Lines GA, HE, being parallel, cannot meet; but in this *Case* the Problem is indeterminate, and capable of infinite Solutions. For, as before, upon CG, describe a Segment of a Circle capable of the observed Angle $\gamma\epsilon\alpha$, and upon CH, describe a Segment capable of the observed Angle $\gamma\alpha\epsilon$: then through C draw a Line any way, cutting the Circles in A, E, these Points will answer the Question.

Problem 3.] *Four Points, B, C, D, F, or the four Sides of a Quadrilateral, with the Angles comprehended, are given; also there are two Stations A and E, such, that at A, only B, C, E, are visible, and at E, only A, D, F; that is, the Angles BAC, BAE, AED, DEF, are given: To find the Places of the two Points A, E; and consequently the Lengths of the Lines AB, AC, AE, ED, EF.* *Fig.* 68.

Construction.] Upon BC (by 33. 3. *E.*) describe a Segment of a Circle, that may contain an Angle equal to the observed Angle BAC; then from C draw the Chord CM, or a Line cutting the Circle in M, so that the Angle BCM, may be equal to the Supplement of the observed Angle BAE, *i. e.* its Residue to 180 deg. In like manner on D F deſcribe a Segment of a Circle, capable of an Angle equal to the obſerved D E F; and from D draw the Chord D N, ſo that the Angle F D N may be equal to the Supplement of the obſerved Angle A E F; join M N, cutting the two Circles in A, E: I ſay, A, E, are the two Points required.

Demonſtration.] Join A B, A C, E D, E F; then is the Angle M A B = B C M (by 21. 3. *E.*) = Supplement of the obſerved Angle B A E, by *Conſtruction*; therefore the conſtructed Angle B A E, is equal to that which was obſerv'd. Alſo the Angle B A C, of the Segment, is, by Conſtruction of the Segment, equal to the obſerved Angle B A C. In like manner the conſtructed Angles A E F, D E F, are equal to the correſpondent obſerved Angles A E F, and D E F; therefore A, E, are the Points required.

Calculation.] In the Triangle B C M, the Angle B C M, (= Supplement of B A E) and Angle B M C, (= B A C) are given, with the Side B C; thence M C may be found: in like manner D N, in the Triangle D N F, may be found. But the Angle M C D (= B C D — B C M) is known, with its Legs M C, C D; therefore its Baſe M D, and Angle M D C, may be known. Therefore the Angle M D N (= C D F — C D M — E D N) is known, with its Legs M D, D N; thence M N, with the Angles D M N, D N M, will be known. Then the Angle C M A (= D M C + D M N) is known, with the Angle M A C (= M A B + B A C) and M C, before found; therefore M A, and A C, will be known. In like manner in the Triangle E D N, the Angles E, N, with the Side D N, being known, the Sides E N, E D, will be known; therefore A E (= M N — M A — E N) is known. Alſo in the Triangle A B C, the Angle A, with its Sides B C, C A, being known, the Side A B will be known, with the Angle B C A; ſo in the Triangle E F D, the Angle E, with the Sides E D, D F, being known, E F will be found, with the Angle E D F. Laſtly, in the Triangle A C D, the Angle A C D (= B C D — B C A) with its Legs A C, C D, being known, the Side A D will be known; and in like manner E C, in the Triangle E D C.

Note, That in this *Problem*, as alſo in the firſt and ſecond, if the two Stations fall in a right Line with either of the given Objects, the *Locus* of A or E being a Circle, the particular Point of A or E cannot be determined from the Things given.

As to the other *Caſes* of this third *Problem*, wherein A and E may ſhift Places, *i. e.* only D, F, E, may be viſible at A, and only A, B, C, at E; or wherein B, D, E, may be viſible at A, and only C, F, A, at E; or wherein A may be on one ſide of the Quadrilateral, and E on the other; or one of the Stations within the Quadrilateral, and the other without it; I preſume that the *Surveyor* will eaſily direct himſelf, by what has been already ſaid.

The *Solution* of this third *Problem* is general, and ſerves alſo for both the precedent. For ſuppoſe C, D, the ſame Point in the laſt Figure, and it gives the *Solution* of the ſecond *Problem*; but if B, C, be ſuppoſed the ſame Points with D, F, by proceeding as in the laſt, you may directly ſolve the firſt *Problem*.

III. The

III. The Variation of the magnetic Needle is fo commonly known, that I need not infift much on the Explication thereof; 'tis certain that the true folar Meridian, and the Meridian fhewn by a Needle, agree but in very few Places of the World; and this too, but for a little Time (if a Moment) together; the Difference between the true Meridian and magnetic Meridian perpetually varying and changing in all Places, and at all Times; fometimes to the Eaftward, and fometimes to the Weftward.

An Error of Common Surveyors, in comparing Surveys taken at long Intervals of Time with the magnetic Needle, demonftrated; by Mr. William Molyneux. N. 230. *p.* 625. July An. 1697.

On which Account 'tis impoffible to compare two *Surveys* of the fame Place, taken at diftant Times, by magnetic Inftruments (fuch as the Circumferentor, by which the *Down Survey*, or Sir *William Petty*'s *Survey* of *Ireland* was taken) without due Allowance be made for this Variation. To which purpofe, we ought to know the Difference between the magnetic Meridian and true Meridian, at that time of the *Down Survey*, and the faid Difference at the Time when we make a *New Survey* to compare with the *Down Survey*.

But here I would not be underftood, as if I propofed hereby to fhew, that a Map of the fame Place, taken by magnetic Inftruments at never fo diftant Times, fhould not at one Time give the fame Figure and Contents as at another Time. This certainly it will do moft exactly, the Variation of the Needle having nothing to do either in the Shape or Contents of the *Survey*. All that is affected thereby, is the Bearings of the Lines run by the Chain, and the Boundaries between Neighbours. And how this may caufe a confiderable Error (unlefs due Allowance be made for it) is what I fhall prove moft fully.

In order to which, let us fuppofe that about the Year 1657 (at which time the *Down Survey* was taken) the magnetic Meridian and true Meridian did agree at *Dublin*, or pretty nigh all over *Ireland*; that is to fay, that there was no Variation. And indeed by Experiment it was at that time found, as I am well affured, that at *Dublin* it was hardly half a Degree.

Let us fuppofe, that in the Year 1695 the Variation was 7 Degrees from the North to the Weftward: That it was really fo, I believe, I am pretty well affured, from an Experiment made by *myfelf* with all Diligence. But this is not material; let us now only fuppofe it.

Let A, B, reprefent the Survey of two Town Lands, one in the Poffeffion of A, and the other in Poffeffion of B, taken by the *Down Survey*, *Anno* 1657, when there was no Variation. *Fig.* 69.

Let the Line N S, running through the Point P, be the true Meridian, and confequently the magnetic Meridian alfo at that Time, becaufe of the fuppofed no Variation; and let this Line N S, be alfo the Boundary between the two Town Lands A, and B.

In the Year 1695, when the Variation is 7 Degrees from the North to the Weftward, B having a Map of the *Down Survey*, and being fufpicious that his Neighbour A had encroached on him by a Ditch P Q, employs a Surveyor to enquire into the Matter: The Surveyor finds by his Map, that the Boundary between B and his Neighbour A, run from the Point P, through a Meadow, directly according to the magnetic Meridian S P N; but obferving the Ditch P Q caft up much to the Eaftward of the prefent magnetic Meridian, he concludes that A has encroached upon B, and that the Ditch ought to have been caſt up along the Line P*q*, the Angle QP *q* being an Angle of 7 deg. that is, the preſent Variation of the Needle, and the Line P*q*, the preſent magnetic Meridian; for which Variation not making any Allowance, he poſitively determines that B has all the Land in the Triangle QP*q*, more than he ought to have; and that his Ditch ought to run along the Line P*q*.

'Tis true indeed, if the *Surveyor* go the whole Surround of the Land A, and B, he will find their Figure and Contents exactly agreeable to the Map here expreſſed. But then the Bearings of the Lines are all 7 deg. different from the Bearings in the Map, and they will run in and out upon the adjacent neighbouring Lands, and cauſe endleſs Differences between their Poſſeſſors; as is manifeſt from the Figure: Wherein the prick'd Lines repreſent the Diſagreement in the Bearings of the Lines, protracted from the Point P; and we ſee A encroaching upon his Neighbours on the Weſtward, as he encroaches upon B, and B's Eaſtward Neighbours encroaching on him, and ſo forward and clear round. Whereas by a due Allowance for the Variation of the Needle, all this Confuſion and Diſagreement is avoided, and every thing hits right.

Thus, for Inſtance, in the Caſe before us, knowing that the magnetic Variation has cauſed the preſent magnetic Meridian to fall in the Line *nq*P*s*, 7 deg. from the North to the Weſtward; to reduce this to the magnetic Meridian at the Time of the *Down Survey*, I muſt make the Meridian of my Map to fall 7 deg. to the Eaſtward of my magnetic Meridian, as we ſee the Line PQ falls 7 deg. to the Eaſtward of the Line P*q*.

What is here ſaid on Suppoſition that the Magnet had no Variation at the Time of the firſt Survey taken, and that it had 7 deg. Variation Weſtward at the Time of the ſecond Survey, may eaſily be accommodated to the Suppoſal of any other Variations at the firſt and ſecond Surveys, *mutatis mutandis*; for knowing the Variations we know their Difference; and if we know their Difference, this gives us the Angle QP*q*, by which we reduce them to each other. The beſt Way therefore to make Maps invariable, conſtant, and everlaſting, were for the Surveyors, who uſe magnetic Inſtruments, to make always Allowance for the magnetic Variation, and to protract and lay down Plats by the true Meridian.

Perhaps it may be objected, That Surveys may be taken without magnetic Inſtruments, and that therefore this Error ariſing from the magnetic Variation, and Change of the Bearing of Lines, may be avoided. To which I anſwer, Firſt, That granting a Survey may be taken without magnetic Inſtruments, this is nothing againſt what we have laid down, relating to Surveys that are taken with magnetic Inſtruments, as the *Down Survey* actually was, and moſt *Surveys* at preſent actually are taken therewith. Secondly, Tho' a Survey may be taken truly without magnetic Inſtruments, ſo as to ſhew the exact Angles and Lines of the Plat, and conſequently the true Contents; yet this will not give the true Bearings of the Lines, or ſhew my Poſition in relation to my Neighbours, or other Parts of the Country. This muſt be ſupplied by the Magnet, or ſomething equivalent thereto, as finding a true meridian Line on your Land by celeſtial Obſervation. And I doubt not but the ancient *Ægyptians*, before the Diſcovery of the Magnet, were forced to ſome ſuch Expedient in their

Surveys

Surveys and Applotments of Lands, between Neighbour and Neighbour, after the *Inundations* of the *Nile*; which, we are told, gave the first Original to *Geometry* and *Surveying:* Absolute Necessity and Use having introduced these, as Delight and Diversion introduced *Astronomy* amongst the *Chaldeans.*

And this brings me to another Objection, which may be made against the Instance before laid down: It may be said, That certainly the Surveyor which B employed was very ignorant, who would chuse to judge of the Line PQ rather by its Bearing, than by determining the Point Q by measuring from H and G. To this I answer, What if both the Points H and G were vanished since the *Down Survey* was taken? What if the whole Face of the Country were changed, save only the Point P, and the Line PQ? How shall the Surveyor then judge of the Line PQ, but by its Bearing? That this is no extravagant Supposition, we have an Example in *Ægypt* abovementioned, where the *Nile* lays all flat before it, and so uniformly covers all with Mud, that there is no Distinction. In such a Case your Bearing must certainly help you out; there is no other Way.

But I answer, secondly, To say that the Surveyor might have determined the Point Q by Admeasurement from G and H, or any other adjoining noted Points, as from F, K, I, *&c.* 'tis very true; but then 'tis against our *Supposition.* I am upon shewing an Error that arises from judging of the Line PQ by *magnetic Bearing*; and to tell me that this might be avoided by another Way, is to say nothing. I myself shew how it may be avoided, by allowing for the Variation; but still it is an Error till it be avoided.

But, thirdly, If B's Surveyor do not allow for the Variation of the Needle, he will never exactly determine even the Points G, F, H, K, *&c.* or any other Points in the Plat; but instead thereof, will fall on the Points *g*, *h*, *f*, *k*.

From what has been laid down, we may see the absolute Necessity of allowing for the Variation of the Magnet, in comparing old Surveys with new ones; for want of which, great Disputes may arise between neighbouring Proprietors of Lands: And it were to be wished, that our honourable and learned Judges would take this Matter into their Consideration, whenever any Business of this Kind comes before them.

IV. I have invented a *Level* with a Tube, with Glasses and a Thread, hanging between four Points, with a Weight in a Box so contrived, that as soon as the Instrument is set down, you have your *Point* of Horizon with a great deal of Exactness. I am making another, which playeth on one Steel Point, standing on a Diamond.

A new Level; by Mr. Butterfield. N. 141. *p.* 1026. Sept. An. 1678.

V. *An Account of a Book omitted*; *viz.* The Art of Levelling; by *M. Mariotte.* N. 74. *p.* 2217.

CHAP.

CHAP. III.

OPTICS.

A new Theory about Light and Colours; by Mr. Isaac Newton. N. 80. *p.* 3075. Feb. An. 1672.

I. IN the Year 1666 (at which time I applied myſelf to the Grinding of Optic Glaſſes of other Figures than Spherical) I procured me a Triangular Glaſs-Priſm, to try therewith the celebrated Phænomena of Colours. And in order thereto, having darkened my Chamber, and made a ſmall Hole in my Window-ſhuts, to let in a convenient Quantity of the Sun's Light, I placed my Priſm at its Entrance, that it might be thereby refracted to the oppoſite Wall. It was at firſt a very pleaſing Divertiſement, to view the vivid and intenſe Colours produced thereby; but after a while applying myſelf to conſider them more circumſpectly, I became ſurpriſed to ſee them in an oblong Form; which, according to the received Laws of Refractions, I expected ſhould have been circular. They were terminated at the Sides with ſtreight Lines, but at the Ends the Decay of Light was ſo gradual, that it was difficult to determine juſtly what was their Figure, yet they ſeemed Semicircular.

Comparing the Length of this coloured *Spectrum* with its Breadth, I found it about five Times greater; a Diſproportion ſo extravagant, that it excited me to a more than ordinary Curioſity of examining from whence it might proceed. I could ſcarce think, that the various Thickneſs of the Glaſs, or the Termination with Shadow or Darkneſs, could have any Influence on Light to produce ſuch an Effect; yet I thought it not amiſs, firſt to examine thoſe Circumſtances, and ſo tried what would happen by tranſmitting Light through Parts of the Glaſs of divers Thickneſſes, or through Holes in the Window of divers Bigneſſes, or by ſetting the Priſm without, ſo that the Light might paſs through it, and be refracted, before it was terminated by the Hole: But I found none of thoſe Circumſtances material. The Faſhion of the Colours was in all theſe Caſes the ſame.

Then I ſuſpected, whether by any Unevenneſs in the Glaſs, or other contingent Irregularity, theſe Colours might be thus dilated. And to try this, I took another Priſm like the former, and ſo placed it, that the Light paſſing thro' them both, might be refracted contrary ways, and ſo by the latter returned into that Courſe from which the former had diverted it: For by this means I thought the regular Effects of the firſt Priſm would be deſtroyed by the ſecond Priſm, but the irregular Ones more augmented, by the Multiplicity of Refractions. The Event was, that the Light, which by the firſt Priſm was diffuſed into an oblong Form, was by the ſecond reduced into an orbicular One, with as much Regularity as when it did not at all paſs through them.

N. 83. *p.* 4061. *Fig.* 70.

That this Experiment may be better apprehended; let E G deſign the Window; F, the Hole in it, thro' which the Light arrives at the Priſms; A B C, the firſt Priſm, which refracts the Light towards P T, painting there the

the Colour in an Oblong, and $\alpha\beta\gamma$, the ſecond Priſm, which refracts back again the Rays to Q, where the long Image PT is contracted into a round one. I ſuppoſe the Plane $\alpha\gamma$ parallel to BC, and $\beta\gamma$ to AC, that the Rays may be equally refracted contrary ways in both Priſms. The Priſms alſo muſt be placed very near to one another; for if their Diſtance be ſo great, that the Colours begin to appear in the Light, before its Incidence on the ſecond Priſm, thoſe Colours will not be deſtroyed by the contrary Refractions of that Priſm. And if a *Lens* be placed in the Hole F, or immediately after the Priſms, ſo that its *Focus* be at the Image Q, or PT, the Perimeter of the Image Q, and the ſtraight Sides of the Image PT, will become much better defined than otherwiſe. So that, whatever was the Cauſe of that Length, 'twas not any contingent Irregularity. N. 80. p. 3076. Feb. An. 1672.

I then proceeded to examine more critically, what might be effected by the Difference of the Incidence of Rays coming from divers Parts of the Sun; and to that end, meaſured the ſeveral Lines and Angles belonging to the Image. Its Diſtance from the Hole or Priſm was 22 Feet; its utmoſt Length $13\frac{1}{4}$ Inches; its Breadth $2\frac{5}{8}$; the Diameter of the Hole $\frac{1}{4}$ of an Inch. The Angle which the Rays, tending towards the middle of the Image, made with thoſe Lines, in which they would have proceeded without Refraction, was 44 deg. 56 min. and the Vertical Angle of the Priſm, 63 deg. 12 min. Alſo the Refractions on both Sides the Priſm, that is, of the Incident and Emergent Rays, were, as near as I could make them, equal, and conſequently about 54 deg. 4 min. And the Rays fell perpendicularly upon the Wall. Now ſubducting the Diameter of the Hole from the Length and Breadth of the Image, there remains 13 Inches in the Length, and $2\frac{3}{8}$ the Breadth, comprehended by thoſe Rays, which paſſed through the Center of the ſaid Hole; and conſequently the Angle of the Hole, which that Breadth ſubtended, was about 31 min. anſwerable to the Sun's Diameter; but the Angle which its Length ſubtended, was more than five ſuch Diameters, namely 2 deg. 49 min.

Having made theſe Obſervations, I firſt computed from them the refractive Power of that Glaſs, and found it meaſured by the Ratio of the Sines, 20 to 31; and then by that Ratio I computed the Refractions of two Rays flowing from oppoſite Parts of the Sun's *Diſcus*, ſo as to differ 31 min. in their Obliquity of Incidence, and found that the emergent Rays ſhould have comprehended an Angle of about 31 min. as they did before they were incident.

But becauſe this Computation was founded on the Hypotheſis of the Proportionality of the Sines of Incidence and Refraction, which though by my own Experience I could not imagine to be ſo erroneous, as to make that Angle but 31 min. which in reality was 2 deg. 49 min. yet my Curioſity cauſed me again to take my Priſm: And having placed it at my Window, as before, I obſerved, that by turning it a little about its Axis to and fro, ſo as to vary its Obliquity to the Light, more than an Angle of 4 or 5 Degrees, the Colours were not thereby ſenſibly tranſlated from their Place on the Wall; and conſequently by that Variation of Incidence, the Quantity of Refraction was not ſenſibly varied. By this Experiment therefore, as well as by the former

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Computation, it was evident, that the Difference of the Incidence of Rays, flowing from divers Parts of the Sun, could not make them after Decussation diverge at a sensibly greater Angle, than that at which they before converged; which being, at most, but about 31 or 32 min. there still remained some other Cause to be found out, from whence it could be 2 deg. 49 min.

Then I began to suspect, whether the Rays, after their Trajection through the Prism, did not move in curve Lines, and according to their more or less Curvity tend to divers Parts of the Wall. And it increased my Suspicion, when I remember'd that I had often seen a Tennis-Ball struck with an oblique Racket, describe such a curve Line. For, a circular as well as a progressive Motion being communicated to it by that Stroke, its Parts on that Side, where the Motions conspire, must press and beat the contiguous Air more violently than on the other, and there excite a Reluctancy and Re-action of the Air proportionably greater. And for the same Reason, if the Rays of Light should possibly be globular Bodies, and by their oblique Passage out of one Medium into another acquire a circulating Motion, they ought to feel the greater Resistance from the ambient Æther on that Side where the Motions conspire, and thence be continually bowed to the other. But notwithstanding this plausible Ground of Suspicion, when I came to examine it, I could observe no such Curvity in them. And besides (which was enough for my purpose) I observed, that the Difference betwixt the Length of the Image, and the Diameter of the Hole through which the Light was transmitted, was proportionable to their Distance.

The gradual Removal of these Suspicions at length led me to the *Experimentum Crucis*, which was this; I took two Boards, and placed one of them close *behind the Prism at the Window*, so that the Light might pass through a small Hole, made in it for the purpose, and fall on the other Board, which I placed at about 12 Feet distance, having first made a small Hole in it also for some of that incident Light to pass through. Then I placed another Prism behind this second Board, so that the Light trajected through both the Boards might pass through that also, and be again refracted before it arrived at the Wall. This done, I took the first Prism in my Hand, and turned it to and fro slowly about its Axis, so much as to make the several Parts of the Image, cast on the second Board, successively pass through the Hole in it, that I might observe to what Places on the Wall the second Prism would refract them. And I saw by the Variation of those Places, that the Light, tending to that End of the Image towards which the Refraction of the first Prism was made, did in the second Prism suffer a Refraction considerably greater than the Light tending to the other End. And so the true Cause of the Length of that Image was detected to be no other, than that *Light* is not similar or homogeneal, but consists of *difform Rays, some of which are more refrangible than others*; so that without any Difference in their Incidence on the same Medium, some shall be more *refracted* than others; and therefore that, according to their *particular Degrees of Refrangibility*, they were transmitted through the Prism to divers Parts of the opposite Wall.

Ibid. p. 3081.

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I ſhall now proceed to acquaint you with another more notable *Difformity* in its Rays, wherein the Origin of Colours is unfolded: Concerning which I ſhall lay down the Doctrine firſt; and then, for its Examination, give you an Inſtance or two of the Experiments, as a Specimen of the reſt.

The Doctrine you will find comprehended and illuſtrated in the following Propoſitions.

1. As the Rays of Light differ in Degrees of Refrangibility, ſo they alſo differ in their Diſpoſition to exhibit this or that particular Colour. Colours are not Qualifications of Light, derived from Refractions, or Reflections of natural Bodies (as 'tis generally believed) but original and connate Properties, which in divers Rays are divers. Some Rays are diſpoſed to exhibit a Red Colour, and no other; ſome a Yellow, and no other; ſome a Green, and no other; and ſo of the reſt. Nor are there only Rays proper and particular to the more eminent Colours, but even to all their intermediate Gradations.

2. To the ſame Degree of Refrangibility ever belongs the ſame Colour, and to the ſame Colour ever belongs the ſame Degree of Refrangibility. The leaſt refrangible Rays are all diſpoſed to exhibit a Red Colour; and contrarily, thoſe Rays which are diſpoſed to exhibit a Red Colour, are all the leaſt refrangible: So the moſt refrangible Rays are all diſpoſed to exhibit a deep Violet Colour; and contrarily, thoſe which are apt to exhibit ſuch a Violet Colour, are all the moſt refrangible: And ſo to all the intermediate Colours in a continued Series belong intermediate Degrees of Refrangibility. And this Analogy betwixt Colours and Refrangibility is very preciſe and ſtrict; the Rays always either exactly agreeing in both, or proportionally diſagreeing in both.

3. The Species of Colour, and Degree of Refrangibility proper to any particular Sort of Rays, is not mutable by Refraction, nor by Reflection from natural Bodies, nor by any other Cauſe that I could yet obſerve. When any one Sort of Rays hath been well parted from thoſe of other Kinds, it hath afterwards obſtinately retained its Colour, notwithſtanding my utmoſt Endeavours to change it. I have refracted it with Priſms, and reflected it with Bodies, which in Day-light were of other Colours; I have intercepted it with the coloured Film of Air, interceding two compreſſed Plates of Glaſs, tranſmitted it through coloured Mediums, and through Mediums irradiated with other Sorts of Rays, and diverſly terminated it; and yet could never produce any new Colour out of it. It would by contracting or dilating become more briſk, or faint, and, by the Loſs of many Rays, in ſome Caſes very obſcure and dark; but I could never ſee it changed in Specie.

4. Yet ſeeming Tranſmutations of Colours may be made, where there is any Mixture of divers Sorts of Rays: For in ſuch Mixtures, the component Colours appear not; but, by their mutual allaying each other, conſtitute a middling Colour. And therefore, if by Refraction, or any other of the aforeſaid Cauſes, the difform Rays, latent in ſuch a Mixture, be ſeparated, there ſhall emerge Colours different from the Colour of the Compoſition. Which Colours are not new generated, but only made apparent by being parted; for

if they be again intirely mixed and blended together, they will again compose that Colour, which they did before Separation. And for the same Reason, Transmutations made by the convening of divers Colours are not real; for when the difform Rays are again severed, they will exhibit the very same Colours which they did before they entered the Composition; as you see Blue and Yellow Powders, when finely mixed, appear to the naked Eye, Green; and yet the Colours of the component Corpuscles are not thereby really transmuted, but only blended. For when viewed with a good Microscope, they still appear Blue and Yellow interspersedly.

5. There are therefore two sorts of Colours, the one Original and Simple, the other compounded of these. The original or primary Colours are, Red, Yellow, Green, Blue, and a Violet-Purple, together with Orange, Indico, and an indefinite Variety of intermediate Gradations.

6. The same Colours in Specie with these primary Ones, may be also produced by Composition. For a Mixture of Yellow and Blue makes Green; of Red and Yellow makes Orange; of Orange and Yellowish Green makes Yellow. And in general, if any two Colours be mixed, which in the Series of those generated by the Prism are not too far distant one from another, they by their mutual Alloy compound that Colour, which in the said Series appeareth in the Midway between them. But those which are situated at too great a Distance, do not so. Orange and Indico produce not the intermediate Green, nor Scarlet and Green the intermediate Yellow.

7. But the most surprizing and wonderful Composition was that of Whiteness. There is no one sort of Rays which alone can exhibit this. 'Tis ever compounded; and to its Composition are requisite all the aforesaid primary Colours, mixed in a due Proportion. I have often with Admiration beheld, that all the Colours of the Prism being made to converge, and thereby to be again mixed, as they were in the Light before it was incident upon the Prism, reproduced Light, entirely and perfectly White, and not at all sensibly differing from a direct Light of the Sun, unless when the Glasses I used were not sufficiently clear; for then they would a little incline it to their Colour.

8. Hence therefore it comes to pass, that Whiteness is the usual Colour of Light; for Light is a confused Aggregate of Rays indued with all sorts of Colours, as they were promiscuously darted from the various Parts of luminous Bodies. And of such a confused Aggregate, as I said, is generated Whiteness, if there be a due Proportion of the Ingredients; but if any one predominate, the Light must incline to that Colour; as it happens in the blue Flame of Brimstone; the yellow Flame of a Candle; and the various Colours of the Fixed Stars.

9. These Things considered, the Manner how Colours are produced by the Prism is evident. For, of the Rays, constituting the incident Light, since those which differ in Colour proportionally differ in Refrangibility, they by their unequal Refractions must be severed and dispersed into an oblong Form in an orderly Succession, from the least refracted Scarlet, to the most refracted Violet. And for the same Reason it is, that Objects when looked upon through a Prism, appear coloured. For the difform Rays, by their unequal Re-

Refractions, are made to diverge towards ſeveral Parts of the *Retina*, and there expreſs the Images of things coloured, as in the former Caſe they did the Sun's Image upon a Wall. And by this Inequality of Refractions, they become not only coloured, but alſo very confuſed and indiſtinct.

10. Why the Colours of the Rainbow appear in falling Drops of Rain, is alſo from hence evident. For thoſe Drops which refract the Rays, diſpoſed to appear Purple, in greateſt Quantity to the Spectator's Eye, refract the Rays of other ſorts ſo much leſs, as to make them paſs beſide it; and ſuch are the Drops on the Inſide of the primary Bow, and on the Outſide of the ſecondary or exterior One. So thoſe Drops, which refract in greateſt Plenty the Rays, apt to appear Red, toward the Spectator's Eye, refract thoſe of other ſorts ſo much more, as to make them paſs beſide it; and ſuch are the Drops on the exterior Part of the primary, and interior Part of the ſecondary Bow.

11. The odd Phænomena of an Infuſion of *Lignum Nephriticum*, Leaf-Gold, Fragments of coloured Glaſs, and ſome other tranſparently coloured Bodies, appearing in one Poſition of one Colour, and of another in another, are on theſe Grounds no longer Riddles. For thoſe are Subſtances apt to reflect one ſort of Light, and tranſmit another; as may be ſeen in a dark Room, by illuminating them with ſimilar or uncompounded Light. For then they appear of that Colour only, with which they are illuminated; but yet in one Poſition more vivid and luminous than in another, accordingly as they are diſpoſed more or leſs to reflect or tranſmit the incident Colour.

12. From hence alſo is manifeſt the Reaſon of an unexpected Experiment, which Mr. *Hook*, ſomewhere in his *Micrography*, relates to have made with two Wedge-like tranſparent Veſſels, filled the one with a Red, the other with a Blue Liquor: namely, that though they were ſeverally tranſparent enough, yet both together became opake; for if one tranſmitted only Red, and the other only Blue, no Rays could paſs through both.

13. I might add more Inſtances of this Nature, but I ſhall conclude with this general one: That the Colours of all natural Bodies have no other Origin than this, that they are variouſly qualified to reflect one ſort of Light in greater Plenty than another. And this I have experimented in a dark Room, by illuminating thoſe Bodies with uncompounded Light of divers Colours. For by that means any Body may be made to appear of any Colour. They have there no appropriate Colour, but ever appear of the Colour of the Light caſt upon them; but yet with this Difference, that they are moſt briſk and vivid in the Light of their own Day-light Colour. *Minium* appeareth there of any Colour indifferently, with which it is illuſtrated, but yet moſt luminous in Red; and ſo Biſe appeareth indifferently of any Colour, with which it is illuſtrated, but yet moſt luminous in Blue: and therefore *Minium* reflecteth Rays of any Colour, but moſt copiouſly thoſe endowed with Red, and conſequently when illuſtrated with Day-light, that is, with all ſorts of Rays promiſcuouſly blended, thoſe qualified with Red ſhall abound moſt in the reflected Light, and by their Prevalence cauſe it to appear of that Colour. And for the ſame Reaſon Biſe, reflecting Blue moſt copiouſly, ſhall appear Blue by the Exceſs of thoſe Rays in its reflected Light; and the like of other Bodies. And that this is the en-

tire and adequate Caufe of their Colours, is manifeft, becaufe they have no Power to change or alter the Colours of any fort of Rays incident apart, but put on all Colours indifferently, with which they are enlighten'd.

Thefe things being fo, it can be no longer difputed, whether there be Colours in the Dark, or whether they be the Qualities of the Objects we fee; no, nor perhaps, whether Light be a Body. For, fince Colours are the Qualities of Light, having its Rays for their intire and immediate Subject, how can we think thofe Rays Qualities alfo, unlefs one Quality may be the Subject of, and fuftain another? which, in effect, is to call it Subftance. We fhould not know Bodies for Subftances, were it not for their fenfible Qualities; and the Principal of thofe being now found due to fomething elfe, we have as good Reafon to believe that to be a Subftance alfo.

Befides, Who ever thought any Quality to be a heterogeneous Aggregate, fuch as Light is difcovered to be? But to determine more abfolutely what Light is, after what Manner refracted, and by what Modes or Actions it produceth in our Minds the Phantafms of Colours, is not fo eafy: And I fhall not mingle Conjectures with Certainties.

Reviewing what I have written, I fee the Difcourfe itfelf will lead to divers Experiments fufficient for its Examination: And therefore I fhall not trouble you further than to defcribe one of thofe which I have already infinuated.

In a darkened Room make a Hole in the Shut of a Window, whofe Diameter may conveniently be about a third Part of an Inch, to admit a convenient Quantity of the Sun's Light: And there place a clear and colourlefs Prifm, to refract the entring Light towards the further part of the Room; which, as I faid, will thereby be diffufed into an oblong coloured Image. Then place a Lens of about 3 Foot Radius (fuppofe a broad Object-Glafs of a three Foot Telefcope) at the Diftance of about 4 or 5 Feet from thence, through which all thofe Colours may at once be tranfmitted, and made by its Refraction to convene at a further Diftance of about 10 or 12 Feet. If at that Diftance you intercept this Light with a Sheet of white Paper, you will fee the Colours converted into Whitenefs again by being mingled. But it is requifite that the Prifm and Lens be placed fteddy, and that the Paper, on which the Colours are caft, be moved to and fro; for by fuch Motion you will not only find at what Diftance the Whitenefs is moft perfect, but alfo fee how the Colours gradually convene and vanifh into Whitenefs; and afterwards, having croffed one another in that Place where they compound Whitenefs, are again diffipated and fevered, and in an inverted Order retain the fame Colours which they had before they entred the Compofition. You may alfo fee, that if any of the Colours at the Lens be intercepted, the Whitenefs will be changed into the other Colours. And therefore, that the Compofition of Whitenefs be perfect, Care muft be taken that none of the Colours fall befide the Lens.

Fig. 71. Thus in the Defign of this Experiment, A B C expreffeth the Prifm fet endwife to fight, clofe by the Hole F, of the Window E G. Its vertical Angle A C B may conveniently be about 63 deg. M N defigneth the Lens. Its Breadth 2 $\frac{1}{2}$, or 3 Inches. S F, one of the ftraight Lines, in which *difform Rays* may be conceived to flow fucceffively from the Sun. F P, and F R, two of of thofe Rays unequally refracted, which the Lens makes to converge towards Q, and after Decuffation to diverge again. And H I, the Paper, at divers Diftances, on which the Colours are projected; which in Q conftitute Whitenefs, but are Red and Yellow in R, r, and ϱ, and Blue and Purple in P, p, and π.

If you proceed further to try the Impoffibility of changing any uncompounded Colour (which I have afferted in the third and thirteenth Propofitions) 'tis requifite that the Room be made very dark, left any fcattering Light, mixing with the Colour, difturb and allay it, and render it compound, contrary to the Defign of the Experiment. 'Tis alfo requifite, that there be a perfecter Separation of the Colours, than, after the Manner above defcribed, can be made by the Refraction of one fingle Prifm; and how to make fuch further Separations, will fcarce be difficult to them that confider the difcovered Laws of Refractions. But if Trial fhall be made with Colours not thoroughly feparated, there muft be allowed Changes proportionable to the Mixture. Thus, if compound Yellow Light fall upon the Blue Bife, the Bife will not appear perfectly Yellow, but rather Green; becaufe there are in the yellow Mixture many Rays endued with Green, and Green being lefs remote from the ufual Blue Colour of Bife than Yellow, is the more copioufly reflected by it.

In like manner, if any one of the prifmatic Colours, fuppofe Red, be intercepted, on Defign to try the afferted Impoffibility of reproducing that Colour out of the others which are pretermitted; 'tis neceffary, either that the Colours be very well parted before the Red be intercepted, or that together with the Red, the neighbouring Colours, into which any Red is fecretly difperfed (that is, the Yellow, and perhaps Green too) be intercepted; or elfe, that Allowance be made for the emerging of fo much Red out of the Yellow-Green, as may poffibly have been diffufed, and fcatteringly blended in thofe Colours. And if thefe Things be obferved, the new Production of Red, or any intercepted Colour, will be found impoffible.

Some Experiments propofed in relation to this Theory. N. 83. p. 4059. May, An. 1672.

II. 1. To contract the Beams of the Sun without the Hole of the Window, and to place the Prifm between the Focus of the Lens and the Hole.

2. To cover over both Ends of the Prifm with Paper at feveral Diftances from the Middle; or with moveable Rings, to fee how that will vary or divide the Length of the Figure.

3. To move the Prifm fo, as the End may turn about, the Middle being fteddy.

4. To move the Prifm by fhoving it, till firft the one Side, then the Middle, then the other Side pafs over the Hole, obferving the fame Parallelifm.

Obfervations on this Propofal by Mr. Newton. N. 83. p. 4060. May, An. 1672.

2. I fuppofe the Defign of the *Propofer* of thefe Experiments is, to have their Events expreffed, with fuch Obfervations as may occur concerning them. Touching the *firft*, I have obferved, that the folar Image falling on a Paper placed at the Focus of the Lens, was by the interpofed Prifm drawn out in Length proportional to the Prifm's Refraction or Diftance from that Focus. And the chief Obfervable here, which I remember, was, that the ftreight Edges of the oblong Image were diftincter than they would have been without the Lens.

Con-

Confidering that the Rays coming from the Planet *Venus*, are much lefs inclined one to another, than thofe which come from the oppofite Parts of the Sun's Difk; I once tried an Experiment or two with her Light. And to make it fufficiently ftrong, I found it neceffary to collect it firft by a broad Lens; and then interpofing a Prifm between the Lens and its Focus, at fuch Diftance that all the Light might pafs through the Prifm, I found the Focus, which before appeared like a lucid Point, to be drawn out into a long fplendid Line by the Prifm's Refraction.

Concerning the *fecond* Experiment, I have occafionally obferved, that by covering both Ends of the Prifm with Paper at feveral Diftances from the Middle, the Breadth of the folar Image will be increafed or diminifhed as much, as is the Aperture of the Prifm, without any Variation of the Length: Or, if the Aperture be augmented on all Sides, the Image on all Sides will be fo much and no more augmented.

Of the *third* Experiment I have occafion to fpeak in my Anfwer to another Perfon; where you will find the Effects of two Prifms, in all crofs Pofitions of one to another, defcribed. But if one Prifm alone be turned about, the coloured Image will only be tranflated from Place to Place, defcribing a Circle, or fome other conic Section on the Wall, on which it is projected, without fuffering any Alteration in its Shape, unlefs fuch as may arife from the Obliquity of the Wall, or cafual Change of the Prifm's Obliquity to the Sun's Rays.

The Effect of the *fourth* Experiment I have already infinuated, telling you that Light paffing through Parts of the Prifm of divers Thickneffes, did ftill exhibit the fame Phænomena.

The Genuine Method of examining this Theory; by Mr. Newton. N. 85. *p.* 5004. July, An. 1672.

III. I cannot think it effectual for determining Truth, to examine the feveral Ways by which Phænomena may be explained, unlefs where there can be a perfect Enumeration of all thofe Ways. You know, the proper Method for enquiring after the Properties of Things, is to deduce them from Experiments. And I told, you that the *Theory* which I propounded, was evinced to me, not by inferring *'tis thus, becaufe 'tis not otherwife*; that is, not by deducing it only from a Confutation of contrary Suppofitions, but by deriving it from Experiments concluding pofitively and directly. The Way therefore to examine it, is, by confidering, Whether the Experiments which I propound do prove thofe Parts of the *Theory* to which they are applied, or by profecuting other Experiments which the *Theory* may fuggeft for its Examination. And this I would have done in a due Method; the Laws of Refraction being thoroughly enquired into and determined, before the Nature of Colours be taken into Confideration. It may not be amifs to proceed according to the Series of thefe Queries; which I could wifh were determined by the Event of proper Experiments, declared by thofe that may have the Curiofity to examine them.

1. Whether *Rays*, that are alike incident on the fame Medium, have *unequal Refractions?* And how great are the *Inequalities* of their Refractions at any Incidence?

2. What is *the Law* according to which each Ray is *more or leſs refracted*; whether it be that the ſame Ray is ever refracted according to the *ſame Ratio* of the Sines of Incidence and Refraction; and divers Rays, according to divers Ratio's; or that the Refraction of each Ray is greater or leſs without any certain Rule? That is, whether each Ray have a certain Degree of Refrangibility, according to which its Refraction is performed; or is refracted without that Regularity?

3. Whether *Rays*, which are endued with particular Degrees of *Refrangibility*, when they are by any Means ſeparated, have particular *Colours conſtantly belonging to them*; *viz.* the leaſt Refrangible, Scarlet; the moſt Refrangible, deep Violet; the middle, Sea-Green; and others, other Colours? And on the contrary?

4. Whether *the Colour* of any ſort of *Rays apart* may be changed by *Refraction?*

5. Whether *Colours* by coaleſcing do really *change* one another to produce a *New Colour*, or produce it by *mixing* only?

6. Whether *a due Mixture of Rays*, indued with all variety of Colours, produces *Light* perfectly *like that of the Sun*, and which hath all the ſame Properties, and exhibits the ſame Phænomena?

7. Whether the *component Colours* of each Mixture be really *changed*; or be only *ſeparated*, when from that Mixture various Colours are produced again by Refraction?

8. Whether there be any *other Colours* produced by *Refraction*, than ſuch as ought to reſult from the *Colours* belonging to the *diverſly Refrangible Rays*, by their being ſeparated or mixed by *that Refraction?*

To determine by Experiments theſe and ſuch like Queries, which involve the propounded Theory, ſeems the moſt proper and direct Way to a Concluſion. And therefore I could wiſh all Objections were ſuſpended, taken from Hypotheſes, or any other Heads than theſe two; of ſhewing the Inſufficiency of Experiments to determine theſe Queries, or prove any other Parts of my Theory, by aſſigning the Flaws and Defects in my Concluſions drawn from them; or of producing other Experiments which directly contradict me, if any ſuch may ſeem to occur. For if the Experiments, which I urge, be defective, it cannot be difficult to ſhew the Defects; but if valid, then by proving the Theory they muſt render all Objections invalid.

Animadverſions upon this Theory. by R. P. Ign. Gaſton Pardies. N. 84. *p.* 4087. June, An. 1672.

IV. 1. This ſo extraordinary an Hypotheſis, which overturns the very Foundations of Dioptrics, and makes uſeleſs all the Practice that has hitherto obtained, is intirely built upon that Experiment of the Glaſs Priſm, in which the Rays of Light entering at the Hole of a Window into a dark Room, and then ſtriking againſt the Wall, or received upon a Paper, are not gathered into a round Spot, as Mr. *Newton* ſeemed to expect from the received Laws of Refractions, but appeared to be extended into an oblong Figure. Whence he concluded, that this oblong Figure proceeded from this Cauſe, that ſome of the Rays were refracted more and ſome leſs.

But

But it ſeems to me, that according to the common and received Laws of Dioptrics, that Figure ought not to be round but oblong. For ſince the Rays proceeding from the oppoſite Parts of the ſolar Diſk have a different Inclination in paſſing through the Priſm, they ought to be differently refracted; and ſince the Inclination of ſome is greater than the Inclination of others by at leaſt 30 Minutes, the Refraction alſo muſt become greater.

Therefore the oppoſite Rays, emerging from the other Superficies of the Priſm, diverge and divaricate more, than if all had proceeded without Refraction, or at leaſt had been equally refracted. But that Refraction of the Rays is performed only towards thoſe Parts which may be conceived to be in Plains perpendicular to the Axis of the Priſm: For there is no Inequality of Refraction towards thoſe Parts which are underſtood to be in Plains parallel to the Axis; as may be eaſily demonſtrated. For the two Superficies of the Priſm may be judged as parallel to one another, in reſpect to the Inclination of the Axis, ſince each is parallel to the Axis. But the Refraction through two parallel plain Superficies is reckoned as none, becauſe as much as the Ray is deflected one Way by the firſt Superficies, ſo much it is turned the other Way by the other Superficies. Therefore ſince the Rays of the Sun paſſing from the Hole through the Priſm are not refracted at the Sides, they proceed farther as if no Superficies of the Priſm had intervened, (reſpect only being had, as I ſaid before, to that lateral Divarication;) but ſince the ſame Rays are refracted towards the upper or lower Parts, ſome indeed more and ſome leſs, as being unequally inclined; it is neceſſary that they muſt divaricate more, and therefore muſt be extended into a longer Figure.

Now if a Calculation be rightly made; as the lateral Rays are found by the learned *Newton* in ſuch a Breadth as ſubtends an Arch of 31 Minutes, which Arch anſwers to the Sun's Diameter; ſo I don't at all doubt, but that Height alſo of the Image being found, which ſubtends an Arch of 2°. 49'. is that itſelf which in that Caſe anſwers to the ſame Diameter of the Sun after the unequal Refractions.

Fig. 72. And in reality, ſuppoſing the Priſm to be A B C whoſe Angle A is 60 Degrees, the Ray to be D E, which makes an Angle of 30 Degrees with the Perpendicular E H; I find it at emerging through F G to make with the Perpendicular F I an Angle of 76°. 22'. But ſuppoſing another Ray *d* E, which with the Perpendicular E H makes an Angle of 29°. 30'; I find this, as it emerges through *f g*, to make with the Perpendicular *f i* an Angle of 78°. 45'. Whence thoſe two Rays D E, *d* E, which are ſuppoſed to proceed from oppoſite Parts of the Solar Diſk, make an Angle with each other of 30 Minutes; but the ſame at emerging in the Lines F G, *f g*, ſo diverge, as to conſtitute an Angle of 2°. 23'. with each other. Now if two other Rays were aſſumed, that approach more to the Perpendicular E H, (which make, for inſtance, with the ſame Perpendicular, one an Angle of 29°. 30', the other an Angle of 29°. 0',) then the ſame Rays at their emerging would diverge ſtill more, and would make a greater Angle, even ſometimes more than three Degrees. And moreover that Interval of the refracted Rays is farther increaſed from hence, that the two Rays D E, *d* E, meeting

in

in E, immediately begin to divaricate, and hit upon two ſeparate Points of the other Superficies, that is, in F and *f*. Therefore it is not ſufficient for duly performing the Calculation, from the Length of the Image projected upon the Paper to ſubtract the Magnitude of the Hole of the Window; ſince even ſuppoſing the Hole E to be indiviſible, there would be as it were another broad Hole in the other Superficies, which is F *f*.

As to what he calls *Experimentum Crucis*, to me it ſeems to fall in with the commonly received Rules of Refraction. For, as I have now ſhewn, the Solar Rays, which approaching and converging make an Angle of 30 Degrees, after they have paſſed through the indiviſible Foramen, diverge into an Angle of two or three Degrees. Therefore it is no wonder if the ſame Rays, ſingly impinging upon another Priſm which is alſo opened with a ſmall Hole, ſhould be unequally refracted, ſince their Inclination is unequal. Nor is it to the purpoſe that thoſe Rays are raiſed or depreſſed by the Rotation of the firſt Priſm, the ſecond Priſm remaining without Motion, (which yet cannot be done in every Caſe;) or that, the firſt remaining immoveable, the ſecond ſhould be moved, that it may ſucceſſively receive the coloured Rays of the whole Image, and tranſmit them through its own Hole. For in either Way it is neceſſary, that thoſe extreme Rays, that is the Red and Violet, ſhould fall upon the ſecond Priſm at an unequal Angle, and thereby that their Refraction ſhould be unequal, ſo that That of the Violet Rays ſhould be the greater.

Since therefore a manifeſt Cauſe appears, why the Figure of the Rays ſhould be oblong, and that Cauſe ariſes from the very Nature of Refraction; there ſeems no Occaſion to have Recourſe to any other Hypotheſis, or to admit any different Frangibility of the Rays.

What afterwards he has conceived concerning Colours, that indeed commodiouſly follows from the foregoing Hypotheſis; yet even that is obnoxious to ſome Difficulties. For as he affirms, that no Colour at all, but rather Whiteneſs, will appear, when the Rays of all Colours are mixed promiſcuouſly, that does not ſeem conformable to all the Phænomena. Surely the Varieties which are obſerved in the mingling of different Bodies, which are imbued with different Colours, the very ſame are perceived in the mingling of different Rays that are alſo imbued with different Colours. And he well obſerves, that as from a Yellow and Blue Body a Green Colour ariſes, ſo from a Yellow and Blue Ray a Green Colour will alſo ariſe. So that if all the Rays of all Colours were confounded together, it is neceſſary in that Hypotheſis, that the ſame Colour ſhould appear which really appears in a Mixture of all Paints. But now if thoſe Colours, that is, Red and Yellow, together with Blue and Purple, and all others, if there be more, were beaten together and mingled; not White would appear, but a dark ſad Colour. Therefore a like Colour would appear in the ordinary Rays of Light, if it conſiſted of a Mixture of all ſorts of Colours.

Now at firſt Aſpect nothing ſeems more ingenious or more proper, than what he ſays concerning the Experiment of the moſt acute Mr. *Hook*, in which two different Liquors, the one Red the other Blue, and both ſeparately

rately tranſparent, but when mixed together become opaque. This, the learned *Newton* ſays, ariſes from hence, that one of the Liquors can only tranſmit the Red Rays, the other only the Blue; whence being mingled they can tranſmit none. This, I ſay, at firſt ſight ſeems very oppoſite; nevertheleſs it ſhould follow from hence, that a like Opacity would ariſe from the Mixture of any Liquors of different Colours, which yet is not true.

Anſwer'd by Mr. Newton. N. 84. *p.* 4091. June, An. 1672. *Fig.* 73.

2. The Reverend Father *Pardies* makes the Refractions on the different Sides of the Priſm as unequal as he can, whereas I make them equal, both in my Experiments, and in the Calculation founded upon thoſe Experiments. Let A B C be a Section of the Priſm perpendicular to its Axis, F L, K G, two Rays croſſing one another in *x* the middle of the Hole, and falling upon that Priſm at G and L, and let their refracted Rays be G H, L *m*, and again H I and *m n*. And whereas I have ſuppoſed that their Refractions at the Side A C are nearly equal to thoſe at the Side B C; if A C and B C are put equal, the Rays G H and L *m* will have a like Inclination to the Baſe of the Priſm A B; and therefore the Angle C L *m* = Ang. C H G, and Ang. C *m* L = Ang. C G H. Wherefore alſo the Refractions in G and *m* will be equal, as likewiſe in L and H, and ſo Ang. K G A = Ang. *n m* B, and Ang. F L A = Ang. B H I: And therefore the refracted Rays H I and *m n* will have the ſame Inclination to one another as the incident Rays F L and K G. Therefore if the Angle F *x* K be 30 Minutes, or equal to the Sun's Diameter, then alſo the Angle comprehended by H I and *m n* will be 30 Minutes, if the Rays F L and K G are ſuppoſed equally refrangible. But to me, who have tried it, that Angle comes out about 2°. 49′, which the Ray H I exhibiting the utmoſt Violet Colour, contains with the Ray *m n* of a Blue Colour; and therefore it muſt be neceſſarily granted, that thoſe Rays are differently refrangible, or that the Refractions are performed according to an unequal Ratio of the Sines of Incidence and Refraction.

The Reverend Father adds beſides, that for rightly performing the Calculation, it is not ſufficient to ſubtract the Bigneſs of the Hole from the Length of the Image projected on the Paper; becauſe even ſuppoſing the Hole to be indiviſible, there would ſtill be as it were another wide Hole in the poſterior Superficies of the Priſm. Yet this notwithſtanding it ſeems to me, that the Refractions of the Rays, croſſing at the anterior as well as poſterior Surface of the Priſm, may be rightly computed from the Principles laid down. But if the Matter were otherwiſe, the Breadth of the Chaſm in the latter Superficies, which is as a Hole, could hardly make an Error of two Seconds; and in practical Matters I think it not worth while to take Notice of ſuch minute Differences.

The Reverend Father does not contradict that which I called *Experimentum Crucis*, while he contends, that the unequal Refractions of the Rays, endued with different Colours, are produced by the unequal Incidences. For when the Rays paſs through two very ſmall, diſtant, and immoveable Holes, thoſe Incidences (as I made the Experiment) were plainly equal, and yet theſe Refractions were very unequal. But if he has any Doubt about my Experiments, I beg that he would try himſelf, and meaſure the Refractions of the Rays endued

dued with different Colours, from equal Incidences, and he will find them to be unequal. If he does not like the Way I made uſe of to try this, (than which I think none can be more ſatisfactory) it is eaſy to contrive others; as I myſelf have experienced not a few, and with good Succeſs.

It is objected to the Theory about Colours, that Powders of different Colours being mixed, they exhibit not a White, but a dark Brown Colour. Now to me White, Black, and all intermediate Brown Colours, which can be compoſed of White and Black mixed, ſeem to differ not ſo much in their Species of Colour as in the Quantity of Light. And in the Mixture of Paints, ſince the ſeveral Corpuſcles reflect only their own Colour, and therefore the greateſt Part of the incident Light is ſuppreſſed and ſtifled; the reflected Light muſt become obſcure, and mixed as it were with Darkneſs, ſo that it cannot exhibit intenſe Whiteneſs, but ſuch as would be cauſed by a Mixture of Black, that is, a dark Brown.

It is then objected, that from any Liquors of different Colours mixed in the ſame Veſſel, as well as being contained in different Veſſels, Opacity ſhould be produced, which yet (he ſays) is not true. But I do not ſee the Conſequence. For very many Liquors act upon one another, and ſecretly induce a new Configuration of Parts upon one another; whence they may become opaque, tranſparent, or endued with various Colours, ſuch as could by no means ariſe from a Mixture of Colours. And for this I always thought, that Experiments of this kind were leſs fit, from whence Concluſions could be drawn. Yet here I will obſerve, that for this Experiment Liquors are required, which are endued with ſtrong and intenſe Colours, which tranſmit but few Rays except thoſe of their own Colour. Such are rarely to be met with, as may be ſeen by illuminating Liquors in a dark Room with the different Colours of the Priſm. For few are found which appear very tranſparent in their own Colours, and opaque in others. It is alſo convenient, that the Colours made uſe of ſhould be oppoſite to one another, ſuch as I think Red and Blue to be, or Yellow and Violet, or alſo Green and that Purple which approaches to Scarlet. And perhaps ſome of theſe Liquors, whoſe tinging Particles will not come together, when mixed may become more opaque. But I am nothing ſolicitous about the Event, as well becauſe the Experiment will be more perſpicuous in Liquors when ſeparate, as becauſe I propoſed that Experiment (as likewiſe that of the Rainbow, of the Tincture of *Lignum Nephriticum*, and the Phænomena of other natural Bodies) not ſo much to prove the Doctrine as to illuſtrate it.

Now as to the Reverend Father's calling my Theory *an Hypotheſis*, I take it not amiſs, becauſe to him it may not yet appear. But I propoſed it with another Intent, and it ſeems to contain nothing elſe but certain Properties of Light, which being now found I think it not difficult to prove; and which if I did not know to be true, I ſhould rather chooſe to reject as vain and empty Speculation, than to own them as my Hypotheſis.

Some farther Objections, by R. P. Pardies. N. 85. p. 5012. Jul. An. 1672.

3. In that Hypotheſis which our *Grimaldus* explains at large, in which it is ſuppoſed that Light is a certain Subſtance moving with a moſt rapid Motion, ſome Diffuſion of Light might be made after paſſing through the Hole,

and the Decussation of the Rays. Also in that Hypothesis, in which Light is supposed to proceed through certain Undulations of subtile Matter, as the very ingenious *Hook* explains; Colours might be accounted for by some kind of Diffusion and Expansion of Undulations, which may be made at the Sides of the Rays beyond the Hole, by Contact itself, and by the Continuation of the Matter. And indeed I have recourse to such an Hypothesis, in the Dissertation concerning the Motion of Undulation, which is the sixth Part of my Mechanics. Where I suppose, that those appearing Colours are produced by that Communication of Motion alone, which is propagated at the Sides by the Undulations going directly on. As if Rays entering at the Hole *a* go on towards *b*, the direct Undulations (having regard to strait and natural Motion) ought to be terminated at the Right Line *a b*; yet because of the Continuity of Matter, there will be some Communication of Motion towards the Sides *e c*, where a certain tremulous and subsultory Succussion will be excited. Now if Colours may be supposed to consist in that lateral subsultory Motion, I imagine all the Phænomena of Colours might be explained, as I have endeavoured to do more at large in the Dissertation now mentioned. From which being supposed it will also appear, why the Breadth of the Colours must be expanded farther than the Divarication of the Rays themselves would require.

As to the *Experimentum Crucis*, I do not at all doubt, but that in his Experiment he makes use of such a Situation, as that there may be an equal Inclination of the incident Rays; since he expresly affirms it was so done by him. But I was not able to guess that from what I had read before, where are supposed two small and very distant Holes, and one Prism near the first Hole which is in the Window; through which Prism the coloured Rays proceeding, fall upon the other distant Hole. It was also added, that in order that all the Rays might fall successively on that second Hole, the first Prism was turned about its Axis. But by this means it must necessarily be, that the Inclination of the Rays falling upon the second Hole must be changed. I also suggested, that the Thing would be the same, whether the first Prism remaining immoveable, the second Hole was raised or depressed, that it might receive successively all the Rays of the painted Solar Image; or, that second Hole remaining immoveable, the first Prism was turned about, so that the same Image might change its Situation, and all the Parts might successively impinge upon the second Hole. But without doubt the most sagacious *Newton* made use of other Cautions.

As to what I had objected about the Colours, I think it very well solved. And that I called his Theory an Hypothesis, I did it without any Design, but made use of the Word that first occurred. Therefore I desire him not to think, that I used that Word in any kind of Contempt.

Answer'd by Mr. Newton. *Ibid. p.* 5014. July An. 1672.

4. The Reverend Father says, that it is possible to explain the Length of the Colours, without a different Refrangibility of the different Rays; suppose from the Hypothesis of P. *Grimaldus*, by the Diffusion of Light, which is supposed to be a certain Substance moved with the greatest Rapidity; or by the Hypothesis of our *Hook*, by the Diffusion or Expansion of Undulations,

tions, which he pretends are excited in the Æther by lucid Bodies, and to be propagated every Way. To which I add, that by the Hypothesis of *Cartesius* may also be imagined a like Diffusion or Endeavour of the Pressure of the Globules, in like Manner as is supposed in the Explication of the Tail of a Comet. And the same Diffusion or Expansion may be feigned according to any other Hypothesis, in which Light is supposed to be a Force, Action, Quality, or Substance of any kind, emitted on all Sides from luminous Bodies.

That I may answer this, it is to be observed, that the Doctrine I have explained concerning Refractions and Colours, consists only in certain Properties of Light, neglecting all Hypotheses by which those Properties may be explained. Wherefore I thought proper here wholly to refrain from the Consideration of Hypotheses, as being no fit Place of Argumentation; and I will abstract the Force of the Objection, that it may receive a fuller and more general Answer.

Therefore by Light I understand any Being or Power of a Being, (whether it be a Substance, or any Power, Action, or Quality of it) which proceeding directly from a lucid Body is adapted to excite Vision. And by Rays of Light I understand the least Parts of it, or any indefinitely little Parts of it, which do not depend on one another; such as are all those Rays which luminous Bodies emit according to Right Lines, either at the same Time or successively. For those as well collateral as successive Parts of Light are independent, since some without others may be intercepted, and may be separately reflected or refracted any Way. And this being allowed, all the Force of the Objection consists in this, that the Colours may be stretched out in Length by some Diffusion of the Light beyond the Hole, which does not arise from the unequal Refrangibility of different Rays, or of the independent Parts of Light.

But I have shewn above, that they are not lengthened by any other Means; and that I might confirm the whole in the strictest Manner, I have added that Experiment which now is well known by the Name of *Experimentum Crucis*; of the Conditions of which since the Reverend Father has had some Doubt, I have thought fit to represent it by a Scheme. Let B C be the *Fig. 75.* former little Board to which the Prism A is immediately prefixed, and let D E be the latter Board, at the Distance of about 12 Feet from the former, behind which the other Prism F is fix'd. But let the Boards be so perforated at *x* and *y*, that a little of the Light refracted by the former Prism may be transmitted through each of the Holes to the second Prism, and there again be refracted. Now let the former Prism be converted about its Axis by a reciprocal Motion, and the Colours falling upon the latter Board D E, will be raised and depressed by Turns, by which Means another and another Colour at Pleasure may be successively made to pass through its Hole *y* to the latter Prism, while the other Colours will fall upon the Board. Then you will see, that the Rays which are endued with different Colours will suffer a different Refraction in that latter Prism, because they will arrive to different Places of the opposite Wall, or of any Obstacle G H removed at the Distance

of

of ſome Feet farther. The Violet Rays will come ſuppoſe to H, the Red to G, and intermediate ones to intermediate Places. And yet becauſe of the determinate Poſition of the Holes, the Incidence of the Rays of each Colour paſſing through each Priſm muſt needs be alike. And thus it appears by meaſuring, that the Rays affected with different Colours will have different Laws of Refractions.

But I ſuſpect what it was that led the Reverend Father into a State of Doubt; he ſeems to have placed his firſt Priſm A behind the Board B C, and thus by converting it about its Axis, it is probable that the Inclination of the Rays, which come between the two Holes, may have ſuffered ſome Change becauſe of the intermediate Refraction. But by the Deſcription before explained, that Board ought to be placed after the Priſm, that the Rays may fall directly between the Holes, as may appear from my Words; *I took two Boards, and placed one of them cloſe behind the Priſm at the Window.* And the Deſign of the Experiment requires the ſame Thing.

But farther I ſhall obſerve, that in this Experiment, becauſe of the Refraction of the ſecond Priſm, the coloured Light is far leſs diffuſed, and divaricates leſs, than when it is quite White, ſo that the Image at G or H is almoſt Circular; eſpecially if the Priſms are made parallel, and in a contrary Situation of their Angles, as is repreſented in the Scheme. And farther, if the Diameter of the Hole *y* is equal to the Breadth of the Colours, there will be no Extenſion of the ſame coloured Light in Length; but the Image which is formed by any Colour at G or H will be really Circular; ſuppoſing the Holes to be Circular, and the Refraction of the latter Priſm not to be greater than of the former, the Rays being nearly perpendicular to the Obſtacle. This ſhews the Diffuſion mentioned before to proceed from a certain Law of the Refractions of every kind of Rays, and not from the Contact or Continuity of Matter undulating or moving very ſwiftly, or any the like Cauſe. But why that Image ſhould be Circular in one Caſe, and in others ſomething ſtretched out in Length, and how the Extenſion of the Light into Length may be leſſened at Pleaſure in any Caſe, I ſhall leave to be determined by Geometricians, and then to be compared with Experiment.

After the Properties of Light have been ſufficiently examined by theſe and the like Experiments, by conſidering the Rays as Parts of it either collateral or ſucceſſive, of which we have found by their Independence that they are diſtinct from one another; Hypotheſes are thence to be judged of, and thoſe are to be rejected which cannot be reconciled with the Phænomena. But it is an eaſy Matter to accommodate Hypotheſes to this Doctrine. For if any one has a Mind to defend the *Carteſian* Hypotheſis, he may ſay that the Globules are unequal, or that the Preſſures of the Globules are ſome ſtronger than others, and thence they become differently refrangible, and proper to excite a Senſation of different Colours. And ſo according to Mr. *Hook*'s Hypotheſis, it may be ſaid, that the Undulations of the Æther are ſome greater or denſer than others. And ſo of the reſt. For this ſeems to be the moſt neceſſary Law and Condition of Hypotheſes, in which natural Bodies are ſuppoſed to conſiſt of a Multitude of Corpuſcles cohering together, that from the different Particles of lucid Bodies, or from the different Parts of the ſame Corpuſcle, (according as they differ in Motion, Figure, Bulk, or any other Qualities) unequal Preſſures, Motions, or moved Corpuſcles, may be propagated every Way through the Æther, from which being confuſedly mixed together, Light may be ſuppoſed to be conſtituted. And nothing can be more difficult in theſe Hypotheſes than the contrary Suppoſition.

From the Aperture or Dilatation of the Light in the latter Face of the Priſm, which the Reverend Father ſays is a Hole as it were, it is ſufficient that no ſenſible Error can ariſe, if there is any at all. And if a Calculation is made exactly according to the Obſervations, the Error will be none. For the Diameter of the Hole being ſubtracted from the Length of the Image, a Length will remain which the Image would have, if the Hole before the Priſm were an indiviſible Point, and that notwithſtanding the aforeſaid Dilatation of the Light in the latter Face of the Priſm; as may eaſily be ſhewn. Then from that given Length of the Image, and its Diſtance from the indiviſible Hole, as alſo from the Poſition and Form of the Priſm, and from the Inclination of the incident Rays to it, and from the Angle which the refracted Rays tending to the Middle of the Image make with the incident Rays from the Center of the Sun, all the other Things may be determined. And what determines the Refractions and Poſitions of the Rays are ſufficient to make a true Calculation of thoſe Refractions. But this Matter is not of ſuch Conſequence as to make any Difficulty.

As to the Reverend Father's calling our Doctrine an Hypotheſis, I believe it proceeded from nothing elſe, but that he uſed the Word which firſt occurred to him, for a Cuſtom has prevailed, that whatever is explained in Philoſophy is called an Hypotheſis. And I had no other Deſign in ſhewing my Diſlike to that Word, than to obviate an Appellation, which may miſlead thoſe who deſire to philoſophize the right Way.

5. Mr. *Newton*'s laſt Anſwer to my Objections has intirely ſatisfied me. The laſt Scruple that I had, about the *Experimentum Crucis*, is fully removed. And now I clearly underſtand by his Figure what I did not underſtand before. When the Experiment was performed after his Manner, every Thing ſucceeded, and I had nothing farther to deſire.

To the Satisfaction of P. Ig. Gaſt. Pardies. N. 85. *p.* 5018. July, An. 1672.

V. The Conſideration on my *Theories* conſiſts in aſcribing an Hypotheſis to me, which is not mine; in aſſerting an Hpotheſis, which, as to the principal Parts, is not againſt me; in granting the greateſt Part of my Diſcourſe, if explicated by that Hypotheſis; and in denying ſome Things, the Truth of which would have appeared by an Experimental Examination.

Some Conſiderations upon this Theory; by --- *Anſwer'd by Mr.* Newton. N. 88. *p.* 5086. Nov. An. 1672.

Of theſe Particulars I ſhall diſcourſe in Order. And *firſt* of the Hypotheſis, which is aſcribed to me in theſe Words: *But grant this firſt Suppoſition, that Light is a Body, and that as many Colours or Degrees as there may be, ſo many Bodies there may be; all which compounded together would make White*, &c. This, it ſeems, is taken for my Hypotheſis. 'Tis true, that from my *Theory* I argue the Corporeity of Light; but I do it without any abſolute Poſitiveneſs, as the Word *perhaps* intimates; and make it at moſt but a very plauſible

plausible Consequence of the Doctrine, and not a fundamental Supposition, nor so much as any Part of it; which was wholly comprehended in the precedent Propositions. And I somewhat wonder, how the Objector could imagine, that, when I had asserted the *Theory* with the greatest Rigour, I should be so forgetful as afterwards to assert the fundamental Supposition itself with no more than a *perhaps*. Had I intended any such Hypothesis, I should somewhere have explained it: But I knew, that the Properties, which I declared of Light, were in some measure capable of being explicated, not only by that, but by many other mechanical Hypotheses; and therefore I chose to decline them all, and to speak of Light in general Terms, considering it abstractly, as something or other propagated every Way in streight Lines from luminous Bodies, without determining what that Thing is; whether a confused Mixture of difform Powers, or Beings whatsoever. And for the same Reason I chose to speak of Colours according to the Information of our Senses, as if they were Qualities of Light without us. Whereas by that Hypothesis, I must have considered them rather as Modes of Sensation, excited in the Mind by various Motions, Figures, or Sizes of the Corpuscles of Light, making various mechanical Impressions on the Organ of Sense: as I expressed it in that Place, where I spake of the Corporeity of Light.

But supposing I had propounded that Hypothesis, I understand not why the Objector should so much endeavour to oppose it: For certainly it has a much greater Affinity with his own Hypothesis, than he seems to be aware of; the Vibrations of the Æther being as useful and necessary in this, as in his; for assuming the Rays of Light to be small Bodies, emitted every Way from shining Substances, those, when they impinge on any refracting or reflecting Superficies, must as necessarily excite Vibrations in the Æther, as Stones do in Water when thrown into it. And supposing these Vibrations to be of several Depths or Thicknesses, accordingly as they are excited by the said corpuscular Rays of various Sizes and Velocities; of what Use they will be for explicating the Manner of Reflection or Refraction, the Production of Heat by the Sun-Beams, the Emission of Light from Burning, Putrifying, or other Substances, whose Parts are vehemently agitated, the Phænomena of thin transparent Plates and Bubbles, and of all natural Bodies, the Manner of Vision, and the Difference of Colours, as also their Harmony and Discord; I shall leave to their Consideration, who may think it worth their Endeavour, to apply this Hypothesis to the Solution of Phænomena.

In the *second* Place, I told you, That the Objector's Hypothesis, as to the fundamental Part of it, is not against me. That fundamental Supposition is; *That the Parts of Bodies, when briskly agitated, do excite Vibrations in the Æther, which are propagated every Way from those Bodies in streight Lines, and cause a Sensation of Light by beating and dashing against the Bottom of the Eye, something after the Manner that Vibrations in the Air cause a Sensation of Sound by beating against the Organs of Hearing.* Now, the most free and natural Application of this Hypothesis to the Solution of Phænomena, I take to be this: That the agitated Parts of Bodies, according to their several Sizes, Figures, and Motions, do excite Vibrations in the Æther of various Depths or

Bignesses,

Bigneſſes, which being promiſcuouſly propagated through that Medium to our Eyes, effect in us a Senſation of Light of a white Colour; but if by any means thoſe of unequal Bigneſſes be ſeparated from one another, the largeſt beget a Senſation of a red Colour; the leaſt or ſhorteſt, of a deep Violet; and the intermediate ones, of intermediate Colours; much after the manner that Bodies, according to their ſeveral Sizes, Shapes, and Motions, excite Vibrations in the Air of various Bigneſſes, which, according to thoſe Bigneſſes, make ſeveral Tones in Sound: That the largeſt Vibrations are beſt able to overcome the Reſiſtance of a refracting Superficies, and ſo break through it with leaſt Refraction; whence the Vibrations of ſeveral Bigneſſes, that is, the Rays of ſeveral Colours, which are blended together in Light, muſt be parted from one another by Refraction, and ſo cauſe the Phænomena of Priſms, and other refracting Subſtances: And that it depends upon the Thickneſs of a thin tranſparent Plate or Bubble, whether a Vibration ſhall be reflected at its further Superficies, or tranſmitted; ſo that, according to the Number of Vibrations interceding the two Superficies, they may be reflected or tranſmitted for many ſucceſſive Thickneſſes. And ſince the Vibrations which make Blue and Violet, are ſuppoſed ſhorter than thoſe which make Red and Yellow, they muſt be reflected at a leſs Thickneſs of the Plate: Which is ſufficient to explicate all the ordinary Phænomena of thoſe Plates or Bubbles, and alſo of all natural Bodies, whoſe Parts are like ſo many Fragments of ſuch Plates.

Theſe ſeem to be the moſt plain, genuine, and neceſſary Conditions of this Hypotheſis. And they agree ſo juſtly with my Theory, that if the Animadverſor think fit to apply them, he need not, on that Account, apprehend a Divorce from it. But yet how he will defend it from other Difficulties, I know not; for to me the fundamental Suppoſition itſelf ſeems impoſſible, namely, that the Waves or Vibrations of any Fluid, can, like the Rays of Light, be propagated in ſtreight Lines, without a continual and very extravagant ſpreading and bending every Way into the quieſcent Medium, where they are terminated by it. I miſtake, if there be not both Experiment and Demonſtration to the contrary. And as to the other two or three Hypotheſes, which he mentions, I had rather believe them ſubject to the like Difficulties, than ſuſpect the Animadverſor ſhould ſelect the worſt for his own.

What I have ſaid of this, may be eaſily applied to all other mechanical Hypotheſes, in which Light is ſuppoſed to be cauſed by any Preſſion or Motion whatſoever, excited in the Æther by the agitated Parts of luminous Bodies; for it ſeems impoſſible that any of thoſe Motions or Preſſions can be propagated in ſtreight Lines, without the like ſpreading every Way into the ſhadowed Medium on which they border. But yet, if any Man can think it poſſible, he muſt at leaſt allow, that thoſe Motions, or Endeavours to Motion, cauſed in the Æther by the ſeveral Parts of any lucid Body, that differ in Size, Figure, and Agitation, muſt neceſſarily be unequal: Which is enough to denominate Light an Aggregate of *Difform Rays*, according to any of thoſe Hypotheſes. And if thoſe original Inequalities may ſuffice to difference the Rays in Colour and Refrangibility, I ſee no reaſon, why they,