A DIFFERENTIAL EQUATION OF CURIOUS SOLUTION

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SUMMARY / ABSTRACT

The authors face the following exercise or initial value problem (IVP) resolution relating to an ordinary differential equation of the first order. The solution of this equation not homogeneous or complete, which apparently seems to be simple, it will be complicated according to the method followed to do so. Computerized developments in series of Taylor and Mc Laurin are efficient to address resolution of the problem.

Key words: differential equation, coefficients, variation of constants, development series, Asymptote, parabolic branch, turning point.

RESUMEN

Los autores afrontan la resolución del siguiente ejercicio o problema de valor inicial (PVI) referente a una ecuación diferencial ordinaria de primer orden. La solución de esta ecuación no homogénea o completa, que aparentemente parece sencilla, se complica según el método que se sigue para ello. Los desarrollos en serie computerizados de Taylor y Mc Laurin resultan eficientes para abordar la resolución del problema planteado.

***Palabras clave**: ecuación diferencial, coeficientes, variación de constantes, desarrollo en serie, asíntota, rama parabólica, punto de inflexión.*

RESUM

Els autors emprenen la resolució del següent exercici o problema de valor inicial (PVI) referent a una equació diferencial ordinària de primer ordre. La solució d'aquesta equació no homogènia o completa, que sembla aparentment simple, es complica segons el mètode emprat. Els desenvolupaments en sèrie computeritzats de Taylor i Mc Laurin són eficients per tal d'abordar la resolució del problema plantejat.

Paraules clau: equació diferencial, coeficients, variació de constants, desenvolupament en sèrie, asímptota, branca parabòlica, punt d'inflexió.

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INTRODUCTION

On the occasion of the elaboration of a monograph published by the Associated Center of the UNED in Tortosa (Franquet, 2013), its author faced the resolution of the following exercise or initial value problem (IVP) referring to the ordinary first-order differential equation:

$$y' + y = \frac{1}{1+x^2}\text{, wih the initial condition: } y(0) = 0.$$

The solution of this non-homogeneous or complete equation, which apparently seems simple, is complicated according to the method followed for this, as we will have occasion to check below.

METHODOLOGY

In effect, the coefficients of the expressed equation are continuous for all x that belongs to the body of the real numbers, that is, that the solution interval is: $\Re \Leftrightarrow -\infty < x < \infty$.

The above differential equation has the form: $y' + p(x)\cdot y = g(x)$, with $p(x) = 1$; in such a way that, to solve it, we find the integrating factor:

$$\mu(x) = \exp\int dx = \exp(x), \Rightarrow \mu(x) = e^x .$$

Now, we multiply the previous ordinary differential equation by said factor $\mu(x) = e^x$, and we will have that:

$$e^x y' + e^x y = e^x/(1+x^2) \Rightarrow (e^x y)' = e^x/(1+x^2) \Rightarrow$$

$\Rightarrow e^x y = \int \left[e^x/(1+x^2)\right]dx + c \Rightarrow y = e^{-x}\int_0^x \left[e^x/(1+x^2)\right]dx + ce^{-x}$, which is the general integral of the problem posed, although it would be more correct to present a more developed result of it.

Of course, we would have reached the same conclusion by direct application of the corresponding formula (Alcaide, 1981), since it is, as we have pointed out, a first order linear ordinary differential equation, or by the method of constant variation. In effect, the equation is of the type:

$$\frac{dy}{dx} + X\cdot y + X_1 = 0\text{, esto es: } \frac{dy}{dx} + y - \frac{1}{1+x^2} = 0\text{, where } X = 1 \text{ y } X_1 = -\frac{1}{1+x^2}.$$

You have to: $\int X\cdot dx = x$; $\int X_1\cdot e^{\int X\cdot dx}\cdot dx = -\int \frac{e^x}{1+x^2}\cdot dx$; from where:

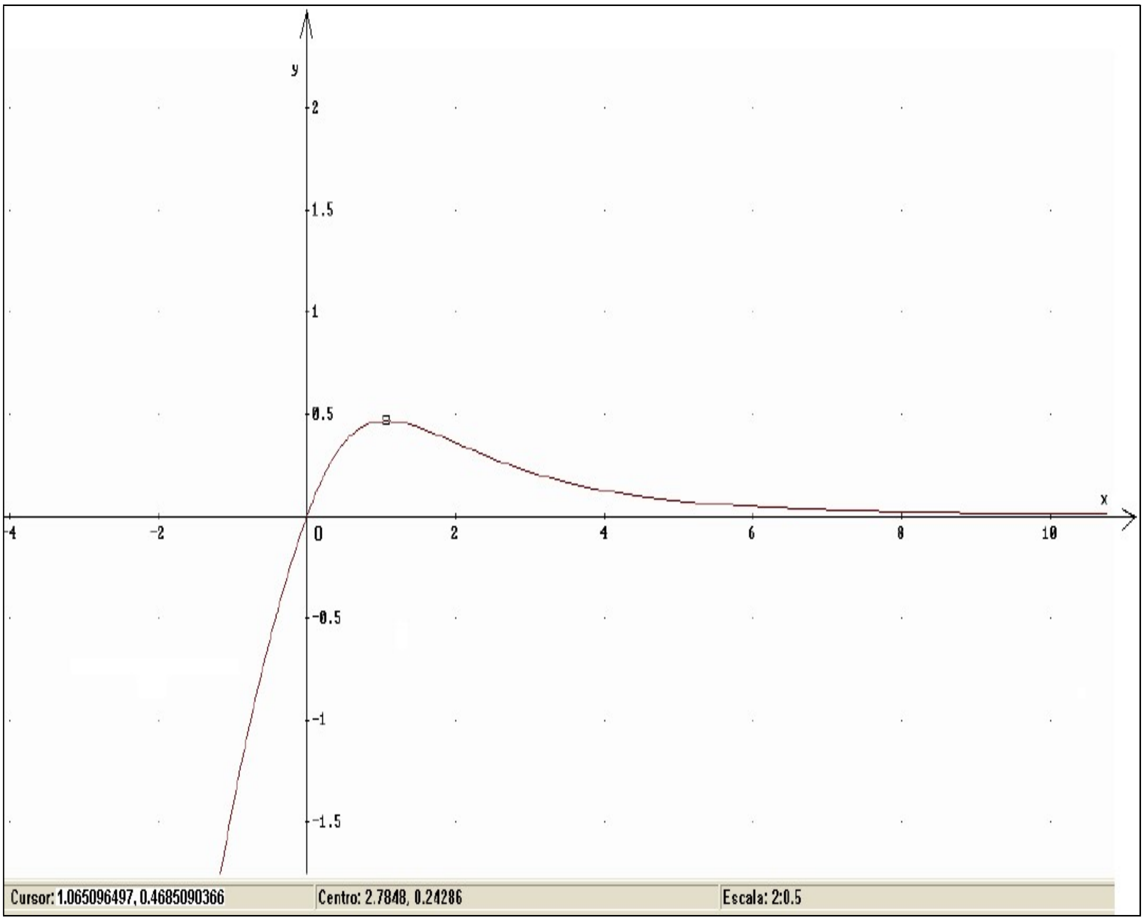
$$y = e^{-x}\cdot\left(c + \int \frac{e^x}{1+x^2}\cdot dx\right) = e^{-x}\int \frac{e^x}{1+x^2}\cdot dx + c\cdot e^{-x}.$$

Substituting now the initial condition given in the previous equation, we obtain:

$$y(0) = 0 = e^0 \int_0^0 \left[e^t/(1+t^2)\right]dt + ce^0 \Leftrightarrow c = 0 .$$

Finally, substituting in the equation, we obtain the particular integral sought:

$\boxed{y = e^{-x}\int_0^x \left[e^x/(1+x^2)\right]\cdot dx}$, whose graphic representation is as follows:



The obtained function passes through the coordinate origin, since when x = 0, it happens that: $y = 1\cdot\int_0^0 \frac{e^x}{1+x^2}\cdot dx = 1\cdot 0 = 0$.

It is evident that there is a horizontal asymptote that is the OX axis itself, since:

$$\lim_{x\to+\infty} y = 0 .$$

On the other hand, when x $\to$ -∞ we will have to:

$$\lim_{x \to -\infty} y = e^{\infty} \cdot \int_0^{-\infty} \frac{e^x}{1+x^2} \cdot dx = -\infty \times \int_{-\infty}^{0} \frac{e^x}{1+x^2} \cdot dx = -\infty,$$

which presumes the existence of parabolic branches, circumstance that will have to be confirmed. To do this, we calculate the expression: $m = \lim_{x \to -\infty} \frac{y}{x} = \lim_{x \to -\infty} \frac{e^{-x} \int_0^x \left[e^x / (1+x^2) \right] \cdot dx}{x}$, limit that does not exist, so we can ensure the absence of parabolic branches.

To find the point at which the function reaches its maximum, we calculate the first derivative (necessary or first degree condition):

$$y' = \frac{1}{1+x^2} - e^{-x} \int_0^x \frac{e^x}{1+x^2} dx .$$

We equal zero and the solution to the equation is x = 1.065096497.

We find the second derivative to check if it is a relative or local maximum (sufficient or second degree condition):

$$y'' = e^{-x} \int_0^x \frac{e^x}{1+x^2} dx - \frac{x^2+2x+1}{\left(1+x^2\right)^2},$$

$$y''(1.065096497) \simeq -0.4675788907 < 0.$$

Then the function peaks (local maximum) at the coordinate point (1.065096497, 0.4685090366).

Now let's look at the turning points. Equalizing the second derivative to zero, we obtain that x = 1.997591819 (Franquet, 2013).

Indeed, it is a turning point since:

$$y'''(1.997591819) = \frac{x^4+2x^3+8x^2+2x-1}{(1+x^2)^3} - e^{-x} \int_0^x \frac{e^x}{1+x^2} dx \Big|_{1.997591819} =$$
$$= 0.1765557587 \neq 0$$

To solve the integral that appears in the previous expression of the particular solution, the following developments of the integrating function must be taken into account $\frac{e^x}{1+x^2}$, as well as the function: $e^{tg\, t}$, in the series of Mc Laurin up to the ninth derivative.

This is, respectively:

$$\begin{cases} f(x)=\dfrac{e^x}{1+x^2}\ ;\ f(0)=1 \\ f'(x)=\dfrac{e^x\left(x^2-2x+1\right)}{\left(1+x^2\right)^2}\ ;\ f'(0)=1 \\ f''(x)=\dfrac{e^x\left(x^4-4x^3+8x^2-4x-1\right)}{\left(1+x^2\right)^3}\ ;\ f''(0)=-1 \\ f'''(x)=\dfrac{e^x\left(x^6-6x^5+21x^4-36x^3+15x^2+18x-5\right)}{\left(1+x^2\right)^4}\ ;\ f'''(0)=-5 \\ f^{IV}(x)=\dfrac{e^x\left(x^8-8x^7+40x^6-120x^5+186x^4-24x^3-224x^2+88x+13\right)}{\left(1+x^2\right)^5};\ f^{IV}(0)=13 \\ f^{V}(x)=\dfrac{e^x\left(x^{10}-10x^9+65x^8-280x^7+770x^6-1020x^5-470x^4+2600x^3-1075x^2-490x+101\right)}{\left(1+x^2\right)^6}; \\ f^{V}(0)=101 \\ f^{VI}(x)=\dfrac{e^x\left(x^{12}-12x^{11}+96x^{10}-540x^9+2145x^8-5400x^7+5480x^6+9960x^5-30045x^4+10980x^3\right)}{\left(1+x^2\right)^7}+ \\ +\dfrac{e^x\left(10980x^3+12216x^2-3852x-389\right)}{\left(1+x^2\right)^7};\ f^{VI}(0)=-389 \end{cases}$$

………………………………………………………… and so on.

From which results the following development (up to the ninth power of x):

$$\frac{e^x}{1+x^2}=f(0)+x\cdot f'(0)\cdot\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{IV}(0)+\frac{x^5}{5!}f^{V}(0)+\frac{x^6}{6!}f^{VI}(0)+\ldots=$$

$$=1+x-\frac{x^2}{2}-\frac{5x^3}{6}+\frac{13x^4}{24}+\frac{101x^5}{120}-\frac{389x^6}{720}-\frac{4241x^7}{5040}+\frac{4357x^8}{8064}+\frac{305353x^9}{362880}-\ldots$$

In the same way we would proceed with the other mentioned development, this is:

$$\begin{cases} f(t) = e^{tg\ t} \text{ ..... ; } f(0) = 1 \\ f'(t) = \dfrac{e^{tg\ t}}{\cos^2 t} \text{ ..... ; } f'(0) = 1 \\ f''(t) = \dfrac{e^{tg\ t}(1 + sen\ 2t)}{\cos^4 t} \text{ ..... ; } f''(0) = 1 \\ f'''(t) = \text{ ..... ; } f'''(0) = 3 \\ f^{IV}(t) = \text{..... ; } f^{IV}(0) = 9 \\ f^{V}(t) = \text{..... ; } f^{V}(0) = 37 \\ f^{VI}(t) = \text{..... ; } f^{VI}(0) = 177 \end{cases}$$

……..... and so on.

From which results the following development (up to the ninth power of t):

$$e^{tg\,t} = f(0) + t\cdot f'(0)\cdot\frac{t^2}{2!}f''(0) + \frac{t^3}{3!}f'''(0) + \frac{t^4}{4!}f^{IV}(0) + \frac{t^5}{5!}f^{V}(0) + \frac{t^6}{6!}f^{VI}(0) + \ldots =$$

$$= 1 + t + \frac{t^2}{2} + \frac{t^3}{2} + \frac{3t^4}{8} + \frac{37t^5}{120} + \frac{59t^6}{240} + \frac{137t^7}{720} + \frac{8711t^8}{5760} + \frac{41641t^9}{362880} + \ldots$$

The sought integral will be like this:

$$\int e^x/(1+x^2)\cdot dx = \int \frac{e^x}{1+x^2}\cdot dx = \boxed{\begin{array}{l} u = e^x \\ v = \text{arctg}\,x \\ dv = \dfrac{dx}{1+x^2} \end{array}} = e^x\cdot\text{arctg}\,x - \int e^x\cdot\text{arctg}\,x\cdot dx\,;$$

this last integral is solved by substitution and, later, by parts, as follows:

$$\int e^x\cdot\text{arctg}\,x\cdot dx = \boxed{\begin{array}{l} x = \text{tg}\,t \\ t = \text{arctg}\,x \\ dv = \dfrac{1}{\cos^2 t} \end{array}} = \int e^{tg\,t}\cdot t\cdot\frac{dt}{\cos^2 t} = \boxed{\begin{array}{l} u = t \\ v = e^{tg\,t} \\ dv = \dfrac{e^{tg\,t}}{\cos^2 t}\cdot dt \end{array}} =$$

$$= t\cdot e^{tg\ t} - \int e^{tg\ t}\cdot dt = t\cdot e^{tg\ t} - \int\left(1 + t + \frac{t^2}{2} + \frac{t^3}{2} + \frac{3t^4}{8} + \frac{37t^5}{120} + \frac{59t^6}{240} + \ldots\right)\cdot dt =$$

$$= t\cdot e^{tg\ t} - t - \frac{t^2}{2} - \frac{t^3}{6} - \frac{t^4}{8} - \frac{3t^5}{40} - \frac{37t^6}{720} - \frac{59t^7}{1680} + \ldots =$$

$$= e^x \cdot \text{arctg}\, x - \text{arctg}\, x - \frac{(\text{arctg}\, x)^2}{2} - \frac{(\text{arctg}\, x)^3}{6} - \frac{(\text{arctg}\, x)^4}{8} - \frac{3\cdot(\text{arctg}\, x)^5}{40} -$$
$$- \frac{37(\text{arctg}\, x)^6}{720} - \frac{59(\text{arctg}\, x)^7}{1680} - \frac{137(\text{arctg}\, x)^8}{5760} - \frac{871(\text{arctg}\, x)^9}{51840} - \ldots$$

whence it follows that:

$$\int_0^x \frac{e^x}{1+x^2}\cdot dx = \text{arctg}\, x + \frac{(\text{arctg}\, x)^2}{2} + \frac{(\text{arctg}\, x)^3}{6} + \frac{(\text{arctg}\, x)^4}{8} + \frac{3\cdot(\text{arctg}\, x)^5}{40} +$$
$$+ \frac{37(\text{arctg}\, x)^6}{720} + \frac{59(\text{arctg}\, x)^7}{1680} + \frac{137(\text{arctg}\, x)^8}{5760} + \frac{871(\text{arctg}\, x)^9}{51840} + \ldots \quad \text{(I)}$$

Otherwise, through alternative development, one would have to:

$$\int_0^x \frac{e^x}{1+x^2}\cdot dx = x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{5x^4}{24} + \frac{13x^5}{120} + \frac{101x^6}{720} - \frac{389x^7}{5040} - \frac{4241x^8}{40320} + \frac{4357x^9}{72576} + \ldots \quad \text{(II)}$$

Apparently, both developments (I) and (II) are different although, obviously, for x = 0 their result is also 0. Let's see what happens in both cases for x = 1 considering only the first seven summand of the development and adjusting up to ten thousandths for the purposes of operational simplification:

Case (I) ⇒ 0.7854 + 0.3084 + 0.0807 + 0.0476 + 0.0224 + 0.0121 + 0.0065 = = 1.2631

Case (II) ⇒ 1.0000 + 0.5000 – 0.1667 – 0.2083 + 0.1083 + 0.1403 – 0.0772 = = 1.2964

Repeating this same process for x = 2, it would be obtained that:

Case (I) ⇒ 1.1071 + 0.6129 + 0.2262 + 0.1878 + 0.1248 + 0.0946 + 0.0716 = = 2.4250

Case (II) ⇒ 2.0000 + 2.0000 – 1.3333 – 3.3333 + 3.4667 + 8.9778 – 9.8794 = = 1.8985

Repeating this same process for x = 3, it would be obtained that:

Case (I) ⇒ 1.2490 + 0.7801 + 0.3248 + 0.3042 + 0.2280 + 0.1951 + 0.1666 = = 3.2478

Case (II) ⇒ 3.0000 + 4.5000 – 4.5000 – 16.8750 + 26.325 + 102.2625 – - 168.7982 = -54.0857

After adding two terms to both series developments in McLaurin and observing that the results of both series are moving away, with the help of the

Derive[3] program the developments of grade 20 are calculated and the resulting integrals are the following:

- With the direct calculation of the Taylor polynomial (case II), the integral turns out to be:

$$\frac{1314502564969066301\cdot x^{21}}{2432902008176640000\cdot 21}-\frac{102360822438075317\cdot x^{20}}{121645100408832000\cdot 20}-\frac{691843455246877\cdot x^{19}}{1280474741145600\cdot 19}+\frac{23023126954133\cdot x^{18}}{27360571392000\cdot 18}+$$

$$\frac{11304631621681\cdot x^{17}}{20922789888000\cdot 17}-\frac{1100370038249\cdot x^{16}}{1307674368000\cdot 16}-\frac{47102631757\cdot x^{15}}{87178291200\cdot 15}+\frac{209594293\cdot x^{14}}{249080832\cdot 14}+\frac{258805669\cdot x^{13}}{479001600\cdot 13}$$

$$-\frac{33588829\cdot x^{12}}{39916800\cdot 12}-\frac{1960649\cdot x^{11}}{3628800\cdot 11}+\frac{305353\cdot x^{10}}{3628800}+\frac{4357\cdot x^{9}}{8064\cdot 9}-\frac{4241\cdot x^{8}}{5040\cdot 8}-\frac{389\cdot x^{7}}{720\cdot 7}+\frac{101\cdot x^{6}}{120\cdot 6}+$$

$$\frac{13\cdot x^{5}}{24\cdot 5}-\frac{5\cdot x^{4}}{24}-\frac{x^{3}}{6}+\frac{x^{2}}{2}+x$$

And for example, for x = 100, take the value $2.530514910\times 10^{40}$.

- With the calculation made by changing the variable and subsequent Taylor polynomial (case I), the integral turns out to be:

$$\frac{23157229065769\cdot \mathrm{ATAN}(x)^{21}}{4742499041280000\cdot 21}+\frac{2832484672207\cdot \mathrm{ATAN}(x)^{20}}{426824913715200\cdot 20}+\frac{8224154352439\cdot \mathrm{ATAN}(x)^{19}}{914624815104000\cdot 19}+$$

$$\frac{4315903789009\cdot \mathrm{ATAN}(x)^{18}}{355687428096000\cdot 18}+\frac{342232522657\cdot \mathrm{ATAN}(x)^{17}}{20922789888000\cdot 17}+\frac{5721418891\cdot \mathrm{ATAN}(x)^{16}}{261534873600\cdot 16}+$$

$$\frac{24362249\cdot \mathrm{ATAN}(x)^{15}}{830269440\cdot 15}+\frac{241586893\cdot \mathrm{ATAN}(x)^{14}}{6227020800\cdot 14}+\frac{35797\cdot \mathrm{ATAN}(x)^{13}}{691200\cdot 13}+\frac{3887\cdot \mathrm{ATAN}(x)^{12}}{57600\cdot 12}+$$

$$\frac{325249\cdot \mathrm{ATAN}(x)^{11}}{3628800\cdot 11}+\frac{41641\cdot \mathrm{ATAN}(x)^{10}}{3628800}+\frac{871\cdot \mathrm{ATAN}(x)^{9}}{5760\cdot 9}+\frac{137\cdot \mathrm{ATAN}(x)^{8}}{720\cdot 8}+\frac{59\cdot \mathrm{ATAN}(x)^{7}}{240\cdot 7}+$$

$$\frac{37\cdot \mathrm{ATAN}(x)^{6}}{720}+\frac{3\cdot \mathrm{ATAN}(x)^{5}}{40}+\frac{\mathrm{ATAN}(x)^{4}}{8}+\frac{\mathrm{ATAN}(x)^{3}}{6}+\frac{\mathrm{ATAN}(x)^{2}}{2}+\mathrm{ATAN}(x)$$

And for example, for x = 100, take the value 28.60279951, and the quotient:

[3] *Derive* is a computational algebra (CAS) program developed by *Texas Instruments*. With it you can carry out a wide range of advanced mathematical calculations as well as represent 2D and 3D graphics in various coordinate systems. It includes the handling of variables, algebraic expressions, equations, functions, vectors, matrices, trigonometry, etc. It also has scientific calculator capabilities. The first version on the market dates from 1988. In the evolution of *Derive* to TI-CAS, it went from being a computer application to being included in the TI-89 and TI-Nspire CAS calculators from *Texas Instruments*. It was also available for Windows and DOS platforms. It was discontinued on June 29, 2007 in favor of TI-Nspire CAS. Its latest version was 6.1 for Windows.

$\frac{28.60279951}{e^{100}} \cong 0$, exemplifies the asymptotic consideration of the point in question much better.

On the other hand, comparing the results that cases (I) and (II) offer for the first integer values of x ($\forall$ x $\in$ {1,2,3}), it is observed, in case (I) and for the first 7 addends of development, which:

$$\begin{cases} \text{For } x = 1 \rightarrow y = e^{-1} \times 1.2631 = 0.46467 \\ \text{For } x = 2 \rightarrow y = e^{-2} \times 2.4250 = 0.32819 \\ \text{For } x = 3 \rightarrow y = e^{-3} \times 3.2478 = 0.16170 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \end{cases}$$

values, all of them, which are better adjusted to the particular integral obtained (note that for x = 3, in case (II), a negative quantity would result), so we will definitely adopt the solution that offers the answer to the problem based on the change of variable and subsequent application of the Taylor polynomial.

Thus, the particular integral sought of the differential equation that is the object of study, will be:

$$y(x) = e^{-x}\int_0^x \frac{e^x}{1+x^2}\cdot dx = e^{-x}[\text{arctg } x + \frac{(\text{arctg } x)^2}{2} + \frac{(\text{arctg } x)^3}{6} + \frac{(\text{arctg } x)^4}{8} + \frac{3\cdot(\text{arctg } x)^5}{40} + \frac{37(\text{arctg } x)^6}{720} + \frac{59(\text{arctg } x)^7}{1680} + \frac{137(\text{arctg } x)^8}{5760} + \frac{871(\text{arctg } x)^9}{51840} + \ldots],$$

with which the problem is definitely solved.

CONCLUSIONS

The solution of this ordinary differential equation of the first order, not homogeneous or complete, with an initial condition, which apparently seems simple, is complicated according to the method followed for this, as we have had occasion to verify, although the computerized serial developments Taylor and Mc Laurin's are highly efficient in addressing the resolution of the problem posed.

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