FORCED PIPELINES:
A THEORETICAL CONTRIBUTION TO THE DEDUCTION OF THE APPROXIMATE FUNCTION OF CHRISTIANSEN

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ABSTRACT / SUMMARY

The estimate of loss of cargo in a pipeline with a discreet cost distribution, a permanent and uniform regimen, a constant flow by derivation and equidistant outlets (conduction under pressure with service in route), was approached and resolved by Christiansen in 1942, in which case the first derivation is situated at the farthest distance upstream of the pipeline equal to the existing space between derivations. Later, in 1957, Jensen and Fratini introduced the corresponding modification in the value of the coefficient by Christiansen, in which case the first outlet of the branch is found at a distance (sprinkling or spray line) equal to half of its space. Such circumstances occur systematically, in the design of watering networks by spray and in those sites of high frequency (micro sprinkling, exudation and dripping). The original study of an academic and engineering nature, which is developed subsequently deals with Christiansen's mathematical justification, and constitutes a theme of notable theoretical interest with little or no diffusion in the existent specialized bibliography in this material.

Key words: pipeline, irrigation, sprinkling, dripping, approach, flow, exits, en-route service, pressure drop, formula.

RESUMEN

El cálculo de las pérdidas de carga en una tubería con distribución discreta del gasto, régimen permanente y uniforme, caudal constante por derivación y salidas equidistantes (conducción a presión con servicio en ruta), fue abordado y resuelto por Christiansen en el año 1942, para el caso en que la primera derivación estuviera situada a una distancia del extremo aguas arriba de la tubería igual al espaciamiento existente entre todas las derivaciones. Posteriormente, en 1957, Jensen y Fratini introdujeron la correspondiente modificación en el valor del coeficiente de Christiansen para el caso de que la primera salida se hallara a una distancia del comienzo del ramal (línea *portagoteros o portaaspersores) igual a la mitad de su espaciamiento. Dichas circunstancias se vienen presentando, sistemáticamente, en el diseño de las redes de riego por aspersión y en los localizados de alta frecuencia (microaspersión, exudación y goteo). El ensayo original, de tipo académico e ingenieril, que se desarrolla a continuación, trata de la justificación matemática de la función aproximada de Christiansen, que constituye un tema de notable interés teórico y de escasa o nula difusión en la bibliografía especializada existente al respecto.*

Palabras clave: *tubería, riego, aspersión, goteo, aproximación, caudal, salidas, servicio en ruta, pérdida de carga, fórmula.*

RESUM

El càlcul de les pèrdues de càrrega en una canonada amb distribució discreta de l'aigua, règim permanent i uniforme, cabal constant per derivació i sortides equidistants (conducció a pressió amb servei en ruta), fou estudiat i resolt per Christiansen l'any 1942, per al cas que la primera derivació es situés a una distància de l'extrem aigües amunt de la canonada igual a l'interval existent entre les mateixes derivacions. Posteriorment, al 1957, Jensen i Fratini introduïren la corresponent modificació en el valor del coeficient de Christiansen per al cas que la primera sortida es trobi a una distància de l'inici de la canonada (línia portagoters o portaaspersors) igual a la meitat d'aquell interval. Les esmentades circumstàncies es presenten, sistemàticament, en el disseny de les xarxes de reg per aspersió i en els localitzats d'alta freqüència (microaspersió, exsudació i degoteig). L'assaig original, de tipus acadèmic i enginyeril, que es desenvolupa a continuació, tracta de la justificació matemàtica de la funció aproximada de Christiansen, la qual cosa constitueix un tema de notable interès teòric i d'escassa o nul·la difusió en la bibliografia especialitzada existent al respecte.

Paraules clau: *canonada, reg, aspersió, degoteig, aproximació, cabal, sortides, servei en ruta, pèrdua de càrrega, fórmula.*

1. INTRODUCTION

In the first issue of the magazine *AGRÓNOMOS* of the Official College of Agricultural Engineers of Levante (corresponding to the summer of 1989), a brilliant collaboration of Teodoro Montalvo López, PhD, Professor of General and Agricultural Hydraulics (Department of Agroforestry Engineering of the Polytechnic University of Valencia) and at that time Director of the Higher Technical School of Agricultural Engineers of the capital of Turia. It addressed there, with commendable clarity and depth, the problem of the generalization of the Christiansen coefficient - used for calculating hydraulic head losses in a pipeline with discrete distribution of expenditure, constant flow rate by bypass and equidistant outlets - for any value of the relationship:

$$r = \frac{l_o}{l}$$

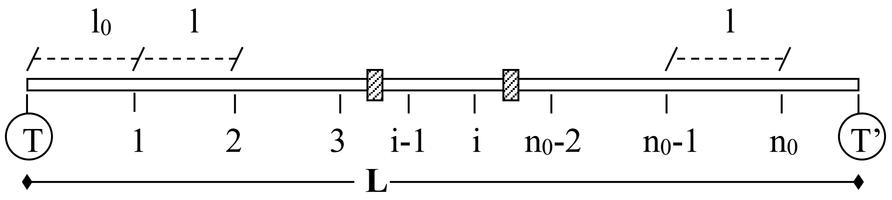
and of the parameters $\mathbf{n_0}$ and $\mathbf{m}$, as well as the direct calculation of the pressure losses in a pipe with a unique characteristic formed by an initial section of any length, in a permanent and uniform regime, and a final section with discrete flow distribution and route service. These circumstances have been systematically presented in the design of sprinkler irrigation networks and in high-frequency localized networks (HFLI, micro-sprinkling, exudation and dripping).

I had the opportunity to contact, more recently, Dr. Montalvo, who proposed to me the study or mathematical deduction of the approximate function of Christiansen, since it is a topic of notable theoretical interest and, apparently, of little or no interest dissemination in the existing specialized bibliography in this regard. In fact, according to Professor Montalvo, the Department of Mathematics of that University had been trying unsuccessfully to date.

The essay or paper that is developed here constitutes, therefore, the mathematical justification (obtained by the subscriber) of the approach of Christiansen's formulation of fruitful applications in the design of modern pressure irrigation systems. It can be considered as a continuation of the articles published by this author in the same magazine *AGRÓNOMOS* (nº: 2, Autumn-Winter 1989/90) and in *ENGINYERIA AGRONÒMICA* (nº: 1, June 1990, of the Official College and the Association of Agricultural Engineers of Catalonia, Spain).

2. JUSTIFICATION OF THE APPROXIMATE FUNCTION

Thus, we have the general case of a pipeline with service in route, with $\mathbf{n_0}$ derivations of constant flow, with a distance between outlets $\mathbf{l}$ and the first derivation being at a distance $\mathbf{l_0}$ from the origin of the conduction, as can be seen in Figure 1:



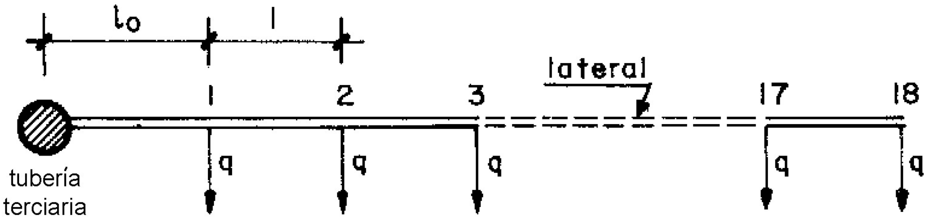


Fig. 1. Pipeline with service in route and equidistant derivations of constant flow q.

In which to be fulfilled $\forall\, l \,/\, l_1 = l_2 = ... = l_i = ... = l$.

Well, the output flow of T, which runs out at T', will be:

$$Q = n_0 \cdot q \ ,$$

and the total length of the pipe, taking into account that: $l_0 = r \cdot l$, is:

$$L = l_0 + (n_0 - 1) \cdot l = (r + n_0 - 1) \cdot l$$

Theoretically, in a pipe of the expressed characteristics, the reduction coefficient for outlets, applicable to the head losses experienced by a single service pipe at its end, would respond to the expression:

$$F = \frac{1}{n_0^{1+m}} \cdot \sum_{i=1}^{n_0} i^m \ ,$$

for which Christiansen (1942) obtained the following approximate function:

$$F = \frac{1}{1+m} + \frac{1}{2 \cdot n_0} + \frac{\sqrt{m-1}}{6 \cdot n_0^2} \text{, meaning:}$$

n_0 = number of derivations or outputs. m = exponent of the formula used in the hydraulic calculation of the head losses.

The problem presented here constitutes a generalization of the classic problem of a simple pipeline with several intermediate intakes (of non-excessive number) and constant diameter, the resolution of which is usually presented by application of the well-known Darcy formula and the prior determination of the line of piezo metric levels.

In the case of equidistant leads and constant flow **q** for each one, the determination of said piezo metric line would be obtained by dividing the total load **h** into parts proportional to the sequence of real numbers: n_0^2, $(n_0 - 1)^2$, ..., 1.

Well, we are going to try, here, to explain or justify mathematically what we will call "Christiansen's approach", basing ourselves, initially, on the classic concept of "integral sum".

Indeed, let's see that the expression: $\sum_{i=1}^{n_0} i^m = 1^m + 2^m + 3^m + ... + n_0^m$,

represents the sum or addition of the areas of the juxtaposed rectangles of heights: 1, 2^m, 3^m, 4^m, 5^m, ..., and of base equal to the unit. See Figure 2:

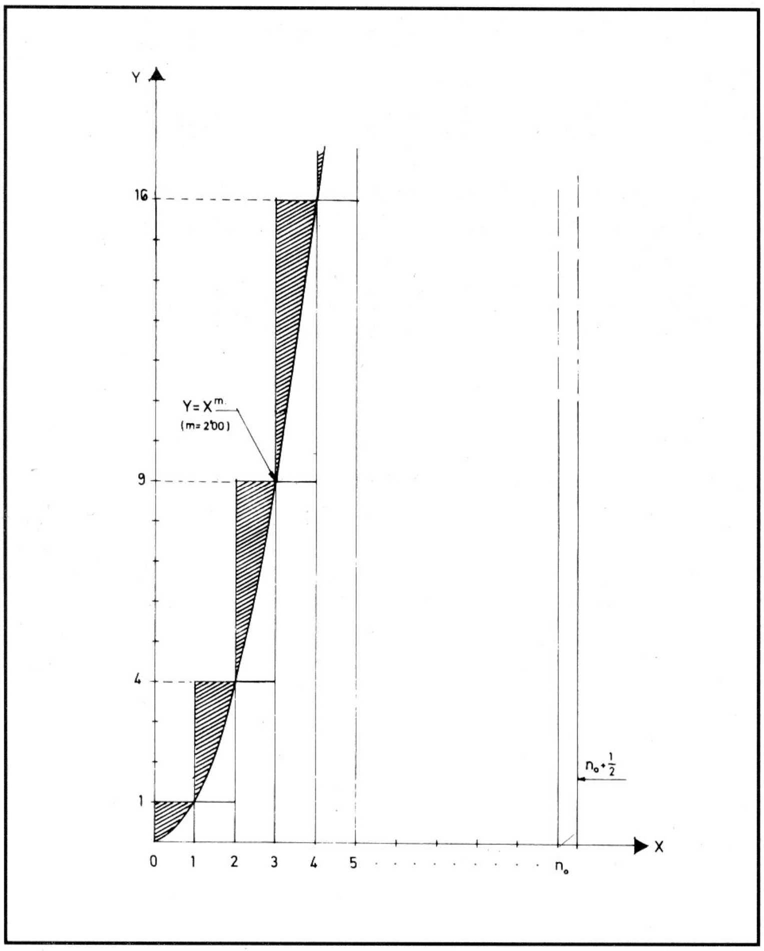


Fig. 2. Graphical representation for m = 2.00.

As can be seen in Figure 2 (performed for m = 2.00), the curve or potential function $y = x^m$, comprises, between it and the axis of abscissa OX, an area that differs from the searched in approximately half the area of the largest rectangle, since, effectively, the area represented in the previous figure by , can be considered equivalent to half the surface of this rectangle.

Likewise, a good approximation to this determination will be obtained taking for the expression: $\sum_{i=1}^{n_0} i^m$, the area under the curve and on the abscissa axis, but between the limits or extreme ordinates:

$$x = 0 \quad \text{and} \quad x = n_0 + \tfrac{1}{2}$$

by the application of the very concept of definite integral. The upper limit will be increased by ½ to obtain precisely half the surface of the larger rectangle, thus:

$$\sum_{i=1}^{n_0} i^m = \int_0^{n_0+\frac{1}{2}} x^m dx = \left[\frac{x^{m+1}}{m+1}\right]_0^{n_0+\frac{1}{2}} = \frac{\left(n_0+\frac{1}{2}\right)^{m+1}}{m+1} = \frac{n_0^{m+1}\cdot\left(1+\frac{1}{2n_0}\right)^{m+1}}{m+1}$$

Now, it is true that:

$$\left(1+\frac{1}{2n_0}\right)^{m+1} = 1+(m+1)\cdot\frac{1}{2n_0}+\frac{(m+1)\cdot m}{2}\cdot\frac{1}{4\cdot n_0^2}+\ldots$$

by the classical formula of the development of the Newton-Tartaglia binomial. The terms that do not appear are third degree and successive in $\frac{1}{n_0}$ and can be neglected, for practical purposes, taking into account their very low magnitude when the number of derivations or outputs $\mathbf{n_0}$ is sufficiently high, as it usually happens in reality.

Thus, the experimental coefficient of reduction for outputs, previously defined, will take the value:

$$F = \frac{1}{n_0^{1+m}}\cdot\sum_{i=1}^{n_0} i^m = \frac{n_0^{m+1}\cdot\left[1+(m+1)\cdot\frac{1}{2n_0}+\frac{(m+1)\cdot m}{2}\cdot\frac{1}{4n_0^2}\right]}{n_0^{m+1}\cdot(m+1)} = \frac{1}{m+1}+\frac{1}{2n_0}+\frac{m}{8n_0^2}$$

With all this, we have already obtained the first two terms or fractional addends of the approximate formula, the deduction of which is the object of our study, namely:

$$\frac{1}{m+1}+\frac{1}{2n_0}$$

Now, the third of them: $\frac{m}{8n_0^2}$ does not coincide with $\frac{\sqrt{m-1}}{6n_0^2}$, that we will find in this formula. Undoubtedly, this is because this third term has been expressly changed or altered (which would be lawful since, ultimately, we are facing a process of approximation) with the sole objective that the formula be valid for particular cases. : m = 1, 2, 3.

Let's see, next, what happens in each of them:

For m = 1, we will have the numerical series:

$$\sum_{i=1}^{n_0} i = \frac{n_0 \cdot (n_0+1)}{2} = 1+2+3+...+n_0,$$

it is the sum of the first non-consecutive terms of an arithmetic progression of ratio equal to unity. So:

$$F = \frac{n_0 \cdot (n_0+1)}{2n_0^2} = \frac{n_0+1}{2n_0} = \frac{1}{2}+\frac{1}{2n_o}$$

At this level, the term $\frac{m}{8\,n_0^2}$ must be changed to another such as, for example, $\frac{m-1}{8\,n_0^2}$, so that 0 is obtained when m = 1.

For m = 2, we will have the numerical series:

$$\sum_{i=1}^{n_0} i^2 = \frac{n_0 \cdot (n_0+1)\cdot(2n_0+1)}{6} = 1+2^2+3^2+...+n_0^2$$

Indeed, this can be demonstrated by induction, since the above equality is obviously true for $n_0 = 1$, since:

$$\frac{1\cdot(1+1)\cdot(2\cdot1+1)}{6} = 1$$

Suppose, also, that it is true for n_0. Then, we will have:

$1^2 + 2^2 + 3^2 + ... + n_0^2 = n_0 \cdot (n_0 + 1)\cdot(2n_0 + 1) / 6$, and adding $(n_0 + 1)^2$ to the two members of the previous expression, the following will result:

$$
\begin{aligned}
1^2+2^2+3^2+\ldots+n_0^2+(n_0+1)^2 &= \frac{n_0\cdot(n_0+1)\cdot(2n_0+1)}{6}+(n_0+1)^2 = \\
&= \frac{(n_0+1)\cdot[n_0\cdot(2n_0+1)+6\cdot(n_0+1)]}{6} = \\
&= \frac{(n_0+1)\cdot[n_0\cdot(2n_0+3)+4n_0+6]}{6} = \\
&= \frac{(n_0+1)\cdot[n_0\cdot(2n_0+3)+2(2n_0+3)]}{6} = \\
&= \frac{(n_0+1)\cdot(n_0+2)\cdot(2n_0+3)}{6}
\end{aligned}
$$

Thus, equality is true for (n_0 + 1), as we intended to demonstrate. Then, the reduction coefficient for outputs will adopt the value:

$$F=\frac{(n_0+1)\cdot(2n_0+1)}{6n_0^2}=\frac{2n_0^2+3n_0+1}{6n_0^2}=\frac{1}{3}+\frac{1}{2n_0}+\frac{1}{6n_0^2}$$

At this level, the term $\frac{m-1}{8n_0^2}$ must be changed to another, such as for example, so that it adopts the value 0 when m = 1, and also applies $\frac{1}{6n_0^2}$, when m = 2.

For m = 3, we will have the numerical series:

$$\sum_{i=1}^{n_0} i^3=\frac{n_0^2\cdot(n_0+1)^2}{4}=1+2^3+3^3+\ldots+n_0^3;$$

in fact, as in the previous case, let us see that this identity is true for: n_0 = 1. Following the same induction method, let us also suppose it true for n_0. Then, it will be fulfilled that:

$1^3 + 2^3 + 3^3 + \ldots + n_0^3 = (1 + 2 + 3 + \ldots + n_0)^2$, and adding $(n_0 + 1)^3$ to the two members of equality, so that it does not vary, will result:

$$
\begin{aligned}
1^3 + 2^3 + 3^3 + \ldots + n_0^3 + (n_0 + 1)^3 &= (1 + 2 + 3 + \ldots + n_0)^2 + (n_0 + 1)\cdot(n_0 + 1)^2 = \\
&= (1 + 2 + 3 + \ldots + n_0)^2 + n_0 (n_0 + 1)^2 + (n_0 + 1)^2.
\end{aligned}
$$

But, as we have seen in the first case (for m = 1), it is true that:

$$n_0(n_0 + 1) = 2 \cdot (1 + 2 + 3 + \ldots + n_0),$$

with which, we will also have to:

$$n_0 (n_0 + 1)^2 = 2\cdot(1 + 2 + 3 + ... + n_0)\cdot(n_0 + 1), \text{ and therefore:}$$

$$1^3 + 2^3 + 3^3 + ... + n_0 + (n_0 + 1)^3 = (1 + 2 + 3 + ... + n_0)^2 + 2\,(1+ 2 + 3 + ... + n_0) \cdot (n_0 + 1) + (n_0 + 1)^2 = [1 + 2 + 3 + ... + n_0 + (n_0 + 1)]^2,$$

which proves that equality is true for $(n_0 +1)$, as we intended to demonstrate. Thus, the reduction coefficient for departures will adopt the value:

$$F = \frac{1}{n_0^4} \cdot \frac{n_0^2 (n_0 + 1)^2}{4} = \frac{n_0^4 + 2n_0^3 + n_0^2}{4n_0^4} = \frac{1}{4} + \frac{1}{2n_0} + \frac{1}{4n_0^2}$$

At this level, the term $\frac{m-1}{6n_0^2}$ must be changed for another that continues to be 0 for m = 1, that is worth $\frac{1}{6n_0^2}$, for m = 2, and that is worth $\frac{1}{4n_0^2}$, when it is m = 3. In the same order of ideas, let us see that its substitution is useful for the term $\frac{\sqrt{m-1}}{6n_0^2}$, since this last expression is valid 0 for m = 1, it is valid $\frac{1}{6n_0^2}$ when m = 2, and, for m = 3 it is not valid $\frac{1}{4n_0^2}$ in the strict sense, but it does take a close value that is: $\frac{\sqrt{2}}{6n_0^2}$, and, $\frac{\sqrt{2}}{6} = 0.2357022$ it is approximately equal to: 1/4 = =0.2500000 (specifically, the first value is 94.28% of the second), which fully satisfies, in fact, our practical requirements.

Since the previously obtained formula is valid for the values of the exponent m = 1, m = 2, m = 3, that is: $m \in (1, 2, 3)$, it will also be valid for non-integer real numbers of the type: $m \in [1, 4]$, that is: 1. ..., 2. ..., 3. ..., and also, although with less degree of approximation, for the assumed values: 4. ..., 5. ..., etc., that could adopt the coefficient used in the formula used in the calculation of the losses of load of the conduction, according to the different cases.

Let us see, with respect to the different values that the coefficient **m** can take, that, in general, the unit pressure losses of a pipe under pressure or forced conduction, depending on the flow through it, respond to a potential expression of the type:

$$J = n \cdot Q^m$$

which, in the case of Darcy's simplified expression, adopts the value: m = 2.00, as well as those of Lèvy, Gaukler, Weisbach, Kütter, Mougnie, Chèzy, Sonier, Manning-Strickler or Catani. In those of the SOGREAH society (1962), Flamant or Blasius is m = 1.75, as well as in those of Saph and Schoder; in that of Scimemi-Veronese, it is m = 1.78571, in that of Hazen-Williams it is m = 1.852, in those of Biegeleisen and Boukowsky it is m = 1.90, as well as in that of

Meyer-Peter, while in the various formulations proposed for Scobey we find the values: m = 1.80, 1.90, etc., but always within the range of existence to which we have referred, and expressing them, all of them, as monomial potential formulas.

The versatility of such a broad formulation leads us to conclude that such information includes the different styles of work successively employed over almost two centuries and, basically, representative of an evolution of knowledge that tends to generalize and unify, each once again, his affirmations, in the pursuit of a final synthesis not yet reached. In this same sense, we have made an effort in the calculation of free conductions, which can be found in other works (Franquet, 2005).

It is unavoidable, nowadays, to distinguish, according to the experimentation of Von Kàrmàn-Nikuradse and Colebrook-White, the smooth, rough and intermediate pipes, these names established not according to the texture of the wall, but according to the hydraulic behavior, by virtue of the configuration of the boundary layer that is perfectly defined in each case. In this way, it happens that the law of resistance in smooth pipes is unique, independent of its constituent material and expressible by an analytical law of which the Blasius formula is a first approximation that has been prolonged by other researchers.

Thus, it will turn out, in short:

$$\boxed{F = \frac{1}{1+m} + \frac{1}{2 \cdot n_0} + \frac{\sqrt{m-1}}{6 \cdot n_0^2}}$$

as it was intended to demonstrate, which is the approximate expression adopted by Christiansen in his study on hydraulic pipes with en-route service.

It should also be borne in mind that this formula will only be valid for the specific case in which the first exit is from the beginning of the conduction at a distance $l_0 = l$ ($r = 1$).

On the other hand, it is obvious that when the number of bypasses or outlets increases indefinitely (that is, when the flow is distributed throughout the entire forced conduction, as in the case of irrigation by exudative or underground tape), the expression above will become:

$$\lim_{n_0 \to \infty} \mathrm{F} = \frac{1}{1+\mathrm{m}},$$

which constitutes, in these circumstances, the minimum value to which the experimental reduction coefficient in question tends. If the residual or extremal flow of the conduction is null, and considering the normal case m = 2.00, let us see that this indicates that the head loss that takes place is one third of that which would occur if the initial flow or expense traveled the entire pipeline and come out freely at the end of it (and considering that the pipeline in question distributes a uniformly distributed expense that is obtained by adding all the expenses of the branches and dividing this sum by the total length of the pipe).

Normally, in HFLI it will be true that m = 1.75, while when the regimen is laminar, this situation is frequent in exudation irrigation in which the head loss is practically continuous and not discrete, we will have that with: m = 1.00 and also F = 0.500 and with m = 2.00 we have F = 0.333.

The most precise study of this case is developed in the following section.

3. EXUDATION PIPING THAT DISTRIBUTES AN UNIFORMLY SPREADED EXPENDITURE OR FLOW

Either a pipeline OB of length **l** and diameter **D**, which originates from a pumping group or from a raised water tank like the one in Figure 3, with several uniformly spaced lateral taps, from which identical expenses are derived. Namely:

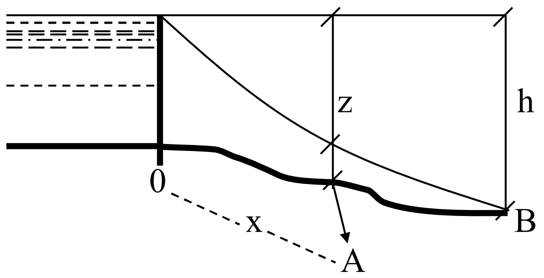


Fig. 3. Outgoing pipe of a tank with intakes of the same flow.

When in a conduction of these characteristics, the number of branches is sufficiently large (typical in sprinkler irrigation systems and localized of high frequency, such as micro sprinkling, exudation and dripping in its different modalities), the calculation is made, with great approximation, assuming that an expense is distributed evenly distributed along the path, which is obtained by adding all the expenses of the derivations and dividing this sum by the total length of the pipe or distance: l = OB. This expense thus obtained is called the expense per unit length of pipe.

In these cases, the movement of the water through the pipe can be assimilated to a succession of infinitesimal uniform movements of variable law with the flow -or with the section of the line if it is not constant- due to the proximity of the changes and the small variation of the flow that takes place as a consequence of them. Although it would be necessary, for the faultless resolution of the problem, the exact knowledge of said law of flow variation, we could admit, with a good approximation, that the service on the route is uniformly distributed throughout the length of the pipe, reducing the flow by a certain quantity **q** per unit length of pipe. **In other words, a flow q per unit of pipe length is spent or consumed** (Torres, 1970).

Now using the following notation:

Q_0 = expenditure in the origin 0 of the pipe.
q = derived expense per unit length of pipe.
Q = available expenditure at a generic point A, located at a distance from the origin OA = x

obviously it will be verified that:

$$Q = Q_0 - q \cdot x \qquad (1)$$

where q·x is the cost distributed in the OA path.

If we express the loss of load due to friction in the OA section, using the formula:

$$z = n\int_0^x Q^2 dx = n\int_0^x (Q_0 - q\cdot x)^2 dx =$$
$$= n\int_0^x (Q_0^2 - 2q\cdot x\cdot Q_0 + q^2\cdot x^2)dx$$

The integration constant is null, since for x = 0, also: z = 0 ⇒ c = 0; that is:

$$z = n(Q_0^2\cdot x - q\cdot Q_0\cdot x^2 + \frac{1}{3}\cdot q^2\cdot x^3) = n\left[(Q+q\cdot x)^2\cdot x - q(Q+q\cdot x)x^2 + \frac{1}{3}\cdot q^2\cdot x^3)\right]$$

$$z = n(Q^2\cdot x + Q\cdot q\cdot x^2 + \frac{1}{3}\cdot q^2\cdot x^3) \qquad (2)$$

which is the equation of a cubic parabola or representative function of the line of piezo metric levels.

If we call Q_e to the residual or extremal flow that comes out of end B of the pipe, we will have, according to equation (2):

$$h = n(Q_e^2\cdot l + Q_e\cdot q\cdot l^2 + \frac{1}{3}\cdot q^2\cdot l^3) \qquad (3)$$

Now, if the end B of the pipe is a dead point, that is, if all the flow is derived along the line without any residual flow reaching point B, it will obviously be that:

$Q_e = 0$ and, therefore, in (1) we will have: $Q_0 = q \cdot l$,

and substituting these values in equation (3), we will obtain:

$$h = \frac{1}{3} \cdot n \cdot q^2 \cdot l^3 = \frac{1}{3} \cdot n \cdot (q \cdot l)^2 \cdot l = 0.333\, n \cdot Q_0^2 \cdot l \quad (4)$$

expression that tells us that the head loss is the third part of what would occur if the cost Q_0 traveled all the pipeline and left freely at the end B of the same, as it has already been stated in the previous section.

Equation (4) can also be expressed like this:

$$h = \frac{1}{3} n \cdot Q_0^2 \cdot l = n \cdot Q'^2 \cdot l$$, from where it results:

$$Q' = \frac{Q_0}{\sqrt{3}} = 0.577 \cdot Q_0 \qquad (5)$$

which means that the head loss is equivalent to that which would occur if a constant flow through the pipeline equals:

$$\frac{Q_0}{\sqrt{3}} = 0.577 \cdot Q_0$$

Next, we will study the procedure used to determine the suitable diameter, so that the pipeline can distribute the cost evenly distributed in the way indicated above.

Equation (3) is equivalent to the formulation:

$$h = n(Q_e^2 + Q_e \cdot q \cdot l + \frac{1}{3} q^2 \cdot l^2) \cdot l = n \cdot Q_1^2 \cdot l = J_1 \cdot l;$$

introducing a dummy flow Q_1 that when circulating through the pipe in a constant way produces a pressure drop *h*.

$$Q_1^2 = Q_e^2 + Q_e \cdot q \cdot l + \frac{1}{3} \cdot q^2 \cdot l^2 \quad (6)$$

but if we consider that:

$$(Q_e + \frac{1}{2} q \cdot l)^2 = Q_e^2 + Q_e \cdot q \cdot l + \frac{1}{4} \cdot q^2 \cdot l^2 < Q_1^2$$

$$(Q_e + \frac{1}{\sqrt{3}} \cdot q \cdot l)^2 = Q_e^2 + \frac{2}{\sqrt{3}} Q_e \cdot q \cdot l + \frac{1}{3} \cdot q^2 \cdot l^2 > Q_1^2$$

Q_1 value bounded between the limits results:

$$Q_e + \frac{1}{2} q \cdot l < Q_1 < Q_e + \frac{1}{\sqrt{3}} \cdot q \cdot l$$, or what is the same:

$$Q_e + 0.5 \cdot q \cdot l < Q_1 < Q_e + 0.577 \cdot q \cdot l\,,$$

then it can be taken with enough approximation, as Q_1 value, to calculate the internal diameter of the pipe:

$$\boxed{Q_1 = Q_e + 0.55\, q \cdot l}$$

which is the formula usually used for the design of agricultural, industrial and domestic water supply networks.

Knowing the values of Q_1 and J = h/l, the value of D and S is easily found. If no flow reaches point B (whereby: $Q_e = 0$), it will be taken, as we have shown, as the value from Q_1 (out of 6):

$$Q_1 = \frac{1}{\sqrt{3}} \cdot q \cdot l \text{ ; or what is the same:}$$

$$\boxed{Q_1 = 0.577 \cdot q \cdot l \approx 58\% \text{ of } Q_0}$$

4. FUNCTION APPROXIMATION

The theoretical general expression that Christiansen tried to simplify, corresponding to the reduction coefficient for n_0 outputs or derivations, as seen in the previous exhibitions, is given by the following formulation, for an exponent of the water velocity **m** given:

$$F = \frac{1}{n_0^{1+m}} \cdot \sum_{i=1}^{n_0} i^m = f(n_0)$$

for which Christiansen (1942), as we have seen, obtained the following approximate function:

$$F = \frac{1}{1+m} + \frac{1}{2 \cdot n_0} + \frac{\sqrt{m-1}}{6 \cdot n_0^2} = g(n_0)\,,$$

where: $m \in [1.75, 2.00]$ for the different formulations usually used in calculating the hydraulic head losses of a pressure pipeline with in-route service, discrete distribution of the flow by equidistant outlets and a permanent and uniform regime.

In short, the problem that Christiansen undoubtedly posed was obtaining the approximation of the function g(x) to the function f(x) with the least possible error, in an environment of the abscissa point: $x = n_0$, or put another way, that given the real function of real variable: F = f(x), defined in $x = n_0$, it was intended to find another real function of real variable: F = g(x) as "simple" as possible and that it "approximates" sufficiently f(x) in a sufficiently small radius environment from the considered point, to the point that at $x = n_0$, it also holds that: $f(n_0) = =g(n_0)$. In this case, the error made in an environment of the point $x = n_0$, when the function g(x) is taken instead of f(x), will be given by the expression:

$$E = \left| f(n_0 + dx) - g(n_0 + dx) \right|$$

On the other hand, the *measure of the approximation* of g(x) to f(x) is a certain number **r**, such that the next limit exists, it is finite and not equal to 0:

$$\lim_{dx \to 0} \frac{E}{dx^r} = \lim_{dx \to 0} \frac{\left| f(n_0 + dx) - g(n_0 + dx) \right|}{dx^r}$$

Somehow the functions that we call "elementals" such as sin x, cos x, log x, e^x, ..., etc., are not really elementary at all; for example, if we want to calculate without x, we find that, except for a few values: $x = 0$, $x = \pi/4$, $x = \pi/2$, ..., etc., the direct calculation of sin x is impossible. However, this is not the case with polynomial functions:

$$f(x) = a_0 + a_1 x + ... + a_n x^n$$

where the operations to be performed are simply arithmetic. Therefore, it is of great interest to obtain formulas that allow the irrational or transcendent functions **to be approximated** by polynomials, in order to calculate their values approximately. Naturally, in every approach, it is necessary to obtain reliable estimates of the error made. Obviously, we cannot expect an exact knowledge of the error, since this would also suppose a precise knowledge of the magnitude that we approximate and would make the approximation unnecessary. What we want, in any case, is **to limit**, so that when making the approach we are sure that the error made does not exceed a certain amount.

Recall that in mathematical analysis, the concept of differential supposes a linear approximation of the function in an environment of the point under consideration. We would say that if f (x) is a differentiable function at point n_0, the affine function g(x) is such that:

$$g(x) = f(n_0) + f'(n_0) \cdot (x - n_0)$$

and approximates the values of f(x) in an environment of n_0. This can be seen graphically in Figure 4.

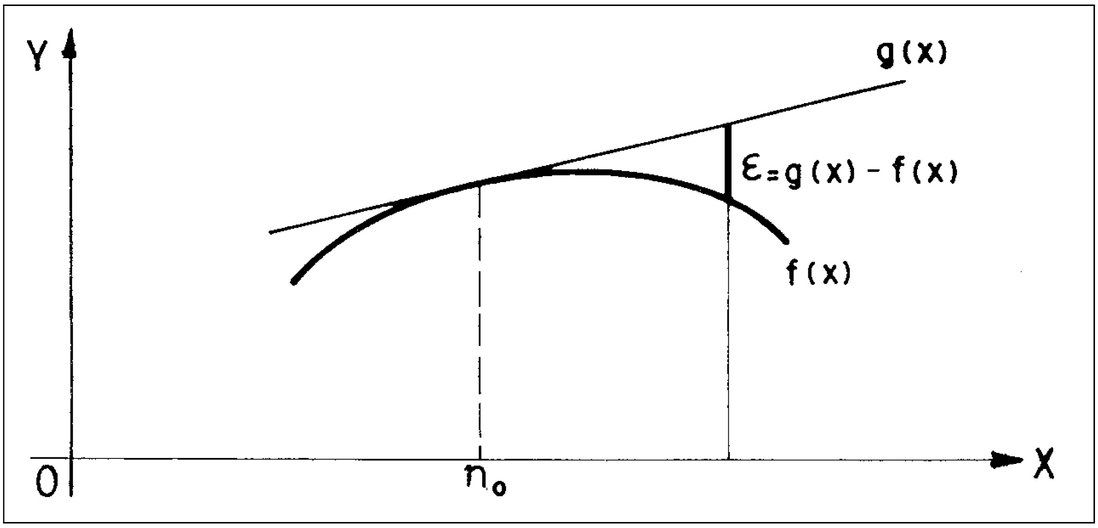


Fig. 4. Approximation between the functions f(x) and g(x) at the point: $x = n_0$.

However, the meaning of the word "approximates", used in the previous statement, is, in our opinion, excessively vague. We can make it more precise if we say that:

$$\lim_{x \to n_0} [g(x) - f(x)] = 0 \quad (7)$$

but, although this equality suggests that f(x) and its approximation g(x) are more and more similar the closer x is to n_0, it does not give us a precise idea of the magnitude of the error made by substituting f(x) for g(x) for a particular value of **x**.

Following this path we can affirm, even more, that:

$$\lim_{x \to n_0} \frac{f(x) - g(x)}{x - n_0} = \lim_{x \to n_0} \left(\frac{f(x) - f(n_0)}{x - n_0} - f'(n_0) \right) = f'(n_0) - f'(n_0) = 0 \quad (8)$$

This statement contained in expression (8), although still imprecise, is stronger than the previous one (7), and guarantees us, not only that the error $|g(x) - f(x)|$ becomes more and more small when approaching n_0, but also that this quantity compared to $(x-n_0)$, which is a magnitude that decreases towards zero, also tends to zero; We will summarize this by saying that $|g(x)-f(x)|$ tends to zero faster than $(x-n_0)$. With symbols, the above statements are expressed by writing:

$$g(x) - f(x) = o(x - n_0)$$

which reads "g(x)-f(x) is an infinitesimal (an infinitely small quantity) compared to (x-n_0)". This notation, which corresponds to Landau's[1] "o", is very useful in calculating limits and for describing terms whose exact expression can be complicated, but whose behavior in the limit is not known to us. To make it more precise, we give the following definition:

"We say that the function h(x) is o($(x-a)^n$), h(x) = ($(x-a)^n$), if it is true that:

$$\lim_{x\to a} \frac{h(x)}{(x - a)^n} = 0 \text{"}$$

Thus, the infinitesimal notation: o($(x-a)^n$) allows us to offer **qualitative** rather than quantitative information about the error made in the functional approach.

On the other hand, we can expect that if a function has several derivatives at one point, it is possible to approximate the values of the function in an environment of that point by functions, rather than linear, polynomial.

At some points on the real line, the approximation of both functions can be total and even coincident with the value taken by f(n_0) and g(n_0). And so, let's see that if we had assumed, for example, an exponent of the velocity of water of m = 2 (if we had used Strickler-Manning's formulation to calculate the head loss of the drip-holder line), with n_0 = 54 equidistant outputs, we would have obtained a theoretical reduction coefficient for outputs of:

$$F = f(n_0) = \frac{(n_0+1)(2n_0+1)}{6n_0^2} = \frac{2n_0^2+3n_0+1}{6n_0^2} = \frac{1}{3} + \frac{1}{2n_0} + \frac{1}{6n_0^2} = \frac{1}{3} + \frac{1}{2{\cdot}54} + \frac{1}{6{\cdot}54^2} = 0.343$$

and, also, the strict application of Christiansen's approximate formula would lead to obtaining the exact same result, since:

$$F = g(n_0) = \frac{1}{1+2} + \frac{1}{2\cdot n_0} + \frac{\sqrt{2-1}}{6\cdot n_0^2} = \frac{1}{3} + \frac{1}{2\cdot 54} + \frac{1}{6\cdot 54^2} = 0.343$$

whereby the error made in the approximation would be null (E = 0).

Recall that, at the beginning of this section, it was said that the intention was to find a certain function g(x) "as simple" as possible and to approach

[1] Given a certain function f(x), with the notation o(f), any function φ(x) is designated such that it is true that: $\lim_{x\to a} \frac{\varphi(x)}{f(x)} = 0$. The previous condition can be replaced by the following: ∀ε > 0, corresponds to an environment: ε*(a) where: $|\varphi(x)| \le \varepsilon |f(x)|$. An equation of the form: φ = o(f) is therefore equivalent to the previous relation.

"sufficiently" the problem function. Previously, we have already indicated how to measure the degree of approximation in question; now, in order not to lose ourselves in subjectivism, what should we understand by the expression "as simple as possible"?

In general, we will take as such the polynomial or parabolic functions (from the 2nd degree), that is, those of analytical configuration of the type:

$$g(x) = a + b \cdot x + c \cdot x^2 + d \cdot x^3 + \ldots$$

whose degree will be determined by the approximation that we wish to obtain and where the constants (a, b, c, d, ...) will be found with the condition that the new function g(x) approximates as closely as possible to f(x).

The simplest approximation, that is, the first degree, is the linear one offered by the equation of the tangent line to the given curve f(x) at the abscissa point $x = n_0$. The higher order approximations can be obtained by applying the well-known Taylor theorem for the development of the function f(x) at that point. In any case, the problem effectively solved by Christiansen became more complex, without, for unknown reasons by the subscriber, said author wanting to publicize, in his day, the mathematical deduction of his famous formula, a question that constitutes, precisely, the fundamental object of this article.

5. GENERALIZED CHRISTIANSEN'S COEFFICIENT

Finally, some other consideration is required. In the particular case that it is true that: $l_0 = ½$ (first outlet located at a distance from the start of the conduction equal to half the space between the other outlets of the pipe), the theoretical general expression: $F = \frac{1}{n_0^{1+m}} \cdot \sum_{i=1}^{n_0} i^m$ it will take the following configuration:

$$F = \frac{1}{n_0^m \cdot \left(n_0 - \frac{1}{2}\right)} \cdot \left(\frac{n_o^m}{2} + \sum_{i=1}^{n_0} i^m\right) \quad ,$$

this expression is due to Jensen and Fratini, which, as already mentioned, will be fulfilled exclusively for the relationship:

$$r = l_0 / l = 1/2.$$

The continuous losses of load in the generic section **i** of the hydraulic axis of the conduction, included between the derivations (**i-1**)-th and **i**-th, are the following:

$$h_i = n \cdot l \cdot Q_i^m$$

and since the circulating flow through section **i** is:

$$Q_i = (n_0 - i + 1)\cdot q \;,$$

load losses in section **i** may also be expressed as:

$$h_i = n \cdot l \cdot (n_0 - i + 1)^m \cdot q^m \;.$$

In this way, the continuous pressure losses throughout the conduction will be:

$$h = \sum_{i=1}^{n_0} h_i = n\cdot q^m \sum_{i=1}^{n_0} l_i \cdot (n_0 - i + 1)^m = n\cdot q^m \left[l_0 \cdot n_0^m + \sum_{i=2}^{n_0} (n_0 - i + 1)^m \right] \;;$$

and how, at the same time, it is true that:

$$\sum_{i=2}^{n_0} (n_0 - i + 1)^m = \sum_{i=1}^{n_0 - 1} i^m \qquad ,$$

the following expression will remain for the head losses:

$$h = n\cdot q^m \cdot l(r\cdot n_0^m + \sum_{i=1}^{n_0-1} i^m) = n\cdot Q^m \cdot L\cdot F$$

Now, taking into account the various relationships above, we obtain:

$$q^m \cdot l(r\cdot n_0^m + \sum_{i=1}^{n_0-1} i^m) = n_0^m \cdot q^m \cdot (r + n_0 - 1)\cdot l\cdot F \quad ;$$

from where:

$$r + \frac{1}{n_0^m} \cdot \sum_{i=1}^{n_0-1} i^m = (r + n_0 - 1)\cdot F \qquad ,$$

whereby solving for the reduction coefficient F (which we will represent by F_r, for any value that the relation **r** may adopt), we have:

$$\boxed{F_r = \frac{r + \frac{1}{n_0^m} \cdot \sum_{i=1}^{n_0-1} i^m}{r + n_0 - 1} \quad (9)}$$

which is the generalized expression of the reduction coefficient for outputs, for any of the values of the parameters **r**, $\mathbf{n_0}$ and **m**.

The values of $F_1(r = 1)$ and $F_{1/2}(r = 1/2)$ have been expressly tabulated below, for different values of $\mathbf{n_0}$ and **m**, namely:

Table 1. Reduction coefficient for outputs F (r = ½).

n_0	m = 1,75	m = 1,80	m = 1,85	m = 1,90	m = 2,00
1	1,000	1,000	1,000	1,000	1,000
2	0,532	0,525	0,518	0,512	0,500
3	0,455	0,448	0,441	0,434	0,422
4	0,426	0,419	0,412	0,405	0,393
5	0,410	0,403	0,397	0,390	0,378
6	0,401	0,394	0,387	0,381	0,369
7	0,395	0,388	0,381	0,375	0,363
8	0,390	0,383	0,377	0,370	0,358
9	0,387	0,380	0,374	0,367	0,355
10	0,384	0,378	0,371	0,365	0,353
11	0,382	0,375	0,369	0,363	0,351
12	0,380	0,374	0,367	0,361	0,349
13	0,379	0,372	0,366	0,360	0,348
14	0,378	0,371	0,365	0,358	0,347
15	0,377	0,370	0,364	0,357	0,346
16	0,376	0,369	0,363	0,357	0,345
17	0,375	0,368	0,362	0,356	0,344
18	0,374	0,368	0,361	0,355	0,343
19	0,374	0,367	0,361	0,355	0,343
20	0,373	0,367	0,360	0,354	0,342
22	0,372	0,366	0,359	0,353	0,341
24	0,372	0,365	0,359	0,352	0,341
26	0,371	0,364	0,358	0,351	0,340
28	0,370	0,364	0,357	0,351	0,340
30	0,370	0,363	0,357	0,350	0,339
35	0,369	0,362	0,356	0,350	0,338
40	0,368	0,362	0,355	0,349	0,338
50	0,367	0,361	0,354	0,348	0,337
100	0,365	0,359	0,353	0,347	0,335
200	0,365	0,358	0,352	0,346	0,334
∞	0,364	0,357	0,351	0,345	0,333

Table 2. Reduction coefficient for outputs F (r = 1).

n_0	m = 1.00	m = 1.75	m = 1.80	m = 1.85	m = 1.90	m = 2.00
1	1.000	1.000	1.000	1.000	1.000	1.000
2	0.750	0.650	0.644	0.639	0.634	0.625
3	0.667	0.546	0.540	0.535	0.528	0.518
4	0.625	0.497	0.491	0.486	0.480	0.469
5	0.600	0.469	0.463	0.457	0.451	0.440
6	0.583	0.451	0.445	0.435	0.433	0.421
7	0.571	0.438	0.432	0.425	0.419	0.408
8	0.563	0.428	0.422	0.415	0.410	0.398
9	0.556	0.421	0.414	0.409	0.402	0.391
10	0.550	0.415	0.409	0.402	0.396	0.385
11	0.545	0.410	0.404	0.397	0.392	0.380
12	0.542	0.406	0.400	0.394	0.388	0.376
13	0.538	0.403	0.396	0.391	0.384	0.373
14	0.536	0.400	0.394	0.387	0.381	0.370
15	0.533	0.397	0.391	0.384	0.379	0.367
16	0.531	0.395	0.389	0.382	0.377	0.365
17	0.529	0.393	0.387	0.380	0.375	0.363
18	0.528	0.392	0.385	0.379	0.373	0.361
19	0.526	0.390	0.384	0.377	0.372	0.360
20	0.525	0.389	0.382	0.376	0.370	0.359
22	0.523	0.387	0.380	0.374	0.368	0.357
24	0.521	0.385	0.378	0.372	0.366	0.355
26	0.519	0.383	0.376	0.370	0.364	0.353
28	0.518	0.382	0.375	0.369	0.363	0.351
30	0.517	0.380	0.374	0.368	0.362	0.350
32	0.516	0.379	0.373	0.367	0.361	0.349
35	0.514	0.378	0.371	0.365	0.359	0.347
40	0.513	0.376	0.370	0.364	0.357	0.345
50	0.510	0.374	0.367	0.361	0.355	0.343
60	0.508	0.372	0.366	0.359	0.353	0.342
80	0.506	0.370	0.363	0.357	0.351	0.340
100	0.505	0.369	0.362	0.356	0.350	0.338
150	0.503	0.367	0.360	0.354	0.348	0.337
300	0.502	0.365	0.359	0.353	0.346	0.335
∞	0.500	0.364	0.357	0.351	0.345	0.333

6. PRACTICAL CALCULATION OF THE CHRISTIANSEN'S UNIVERSAL COEFFICIENT

However, taking into account the infinite number of possible values of **r**, it will be more practical than tabulating the previous equation (9) based on the corresponding values for r = 1 for the calculation of the rest of the values of F. Indeed, given that equality is fulfilled:

$$\sum_{i=1}^{n_0-1} i^m = \sum_{i=1}^{n_0} i^m - n_0^m \quad ,$$

and in addition to the equation: $F = \frac{1}{n_0^{1+m}} \cdot \sum_{i=1}^{n_0} i^m$ it follows that:

$$\sum_{i=1}^{n_0} i^m = F \cdot n_0^{1+m} \quad ,$$

Equality will also have to be satisfied:

$$\sum_{i=1}^{n_0-1} i^m = F \cdot n_0^{1+m} - n_0^m \quad ,$$

which introduced in the expression: $F_r = \dfrac{r + \frac{1}{n_0^m} \cdot \sum_{i=1}^{n_0-1} i^m}{r + n_0 - 1}$, transforms it into:

$$F_r = \frac{r + \frac{1}{n_0^m} \cdot \left(F \cdot n_0^{1+m} - n_0^m\right)}{r + n_0 - 1} = \frac{r + n_0 \cdot F - 1}{r + n_0 - 1} \quad ,$$

that allows obtaining the value of Fr for any value of **r**, depending on the one corresponding to F_1 (r = 1), for the same values of the remaining parameters $\mathbf{n_0}$ and **m**. Obviously, for n_0 = 1, also F_r = 1.00, and this regardless of the relationship values: $r = 1_0/l$.

In short, it should be noted that, in practice, with any value of the parameter **r**, this generalized coefficient allows the direct calculation of load losses in a forced driving with a unique characteristic, variable by an initial section of any length in permanent and uniform or stationary regime and of a final section with discrete distribution of flow and service in route.

7. CONCLUSIONS

The calculation of the losses of load of a conduction of unique characteristic, with service in route and discreet distribution of the expense, permanent and uniform regime, constant flow by derivation and equidistant exits, of great practical interest for the dimensioning of sprinkler irrigation facilities. and localized of high frequency (micro-sprinkling, exudation, dripping) was solved by Christiansen (1942) by means of an approximate formulation whose mathematical justification, which was never exposed by its author, is made in this article, which constitutes a topic of notable interest theoretical and of little or no diffusion in the specialized bibliography existing in this regard.

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