100cias@uned
Facultad de Ciencias
N.º 4 (nueva época) | 2011
ISSN: 1989-7189

Vida Científica

COLABORACIONES EN FÍSICA

THE RELATIONSHIP BETWEEN MATHEMATICS AND PHYSICS

1. INTRODUCTION

James Hopwood Jeans, the English physicist, astronomer and mathematician summarized his feelings on the relationship between mathematics and physics, saying, "we may say that we have already considered with disfavour the possibility of the universe having been planned by a biologist or an engineer; from the intrinsic evidence of his creation, the Great Architect of the Universe now begins to appear as a pure mathematician."[1] The quotation is from Jeans' aptly titled book, *The Mysterious Universe*. Patently the link between mathematics and physics is both deep and intimate, but why should such a link exist?

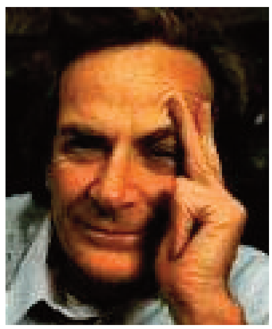
The purpose of this essay is to examine the nature of the link between these two subjects and the interplay between them. We will briefly examine the role that mathematics has played in the development of the physical sciences historically, before looking in more detail at the nature of the relationship between them; many of the arguments and conclusions presented are by no means cut and dried, so we will finish with some speculation about what the future may reveal about the nature of the link from a few well known proponents.

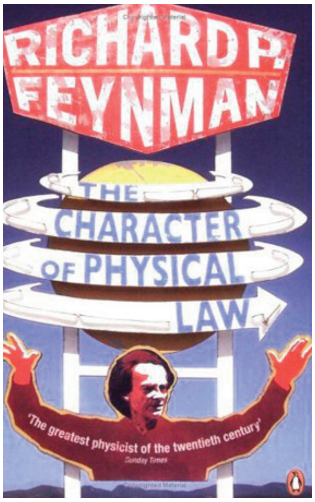
2. THE HISTORICAL ROLE OF MATHEMATICS IN THE PHYSICAL SCIENCES

Even the earliest examples of scientific thought are characterised by their dependence upon mathematics. The first evidence of the recognition of the periodicity of the motions of celestial bodies is a series of Babylonian tablets dating from the 17th century BC; the Babylonians employed a sexigesimal number system which they used to predict these periodic motions [2].

About a millennium later the great Greek natural philosophers began to develop theories about the world based on their own mathematical system. The main difference between the Greek and Babylonian mathematical systems was the axiomatic approach. For example, while the Babylonians knew some rules of geometry and were able in a limited sense to make logical deductions, it was the Greeks, notably Euclid, who realised that all the rules of geometry could be deduced from a simple set of axioms.

The Greeks of course applied their own brand of mathematics to the sciences, an obvious example being Archimedes' proof of the law of the lever. It is interesting to note at this point that in *The Character of Physical Law* Feynman says "The method of always starting from the axioms is not very efficient in obtaining theorems... In physics we need the Babylonian method and not the Euclidean or Greek method." [3] Modern mathematics however is certainly based on an axiomatic approach and this is something that we shall subsequently have to investigate when we examine the nature of the link. Some early Greek schools of thought, notably Pythagoreanism, went as far as to say that numbers and 'number theory' were the basis of the entire physical universe. For example Pythagoras' view is said to have been, "that the cosmos was not only expressible in terms of number, but *was*





number." [4] As Aristotle put it, "[Pythagoreans] considered the first principles of mathematics to be the first principles of all things. Now in mathematics, numbers are naturally first principles" [4]. Despite communicating in writing many of the cosmological ideas of Pythagoreanism, Aristotle was severely critical of the doctrine. It is easy to see why; a whole universe that can be explained by numbers is a somewhat implausible concept. However as we shall see, particularly in the final section of this essay, these ideas may not be as incredible as a first glance suggests.

Similarly mathematics was being used in physical science in China, India and elsewhere. However, the next distinct step in the interplay between maths and physics came in the middle ages. At this stage of history we have many candidates for precursors of the modern scientific method, notably the Muslim scientist Ibn al-Haytham and Roger Bacon, an advocate for empiricism. However, another candidate more pertinent to our discussion is Robert Grosseteste (1175–1253), the English statesman and bishop of Lincoln who emphasised the need for mathematics in order to understand nature. This view was particularly prevalent in his work *De lineis, angulis et figuris* in which he states, "the diligent investigator of natural phenomena can give the causes of all natural effects, therefore, in this way by the rules and roots and foundations given from the power of geometry" [5]. Although we see that mathematics has been used heavily in describing the physical world up to this point, Grosseteste's explicit statement of the reliance of physical theory upon mathematics marks a key step in the development of our understanding of its role.

Galileo Galilei, widely credited as the progenitor of the modern scientific method echoed these sentiments when he said "Philosophy is written in that big book which is continually open in front of our eyes (I mean the universe) which however one cannot understand, if, in advance, one does not master the language and one does not know the ciphers it uses. The language is mathematics and the ciphers are circles, triangles and other geometrical figures." [6] So we have arrived at the era of the modern scientific method, and throughout history to this point, mathematics has played a central role. As we proceed forwards through the work of Newton and Maxwell and on to Relativity, Quantum theory and beyond, the part played by mathematics has, if anything, intensified. As Feynman puts it "what turns out to be true is that, the more we investigate, the more laws we find, and the deeper we penetrate nature, the more [our need for mathematics] persists. Every one of our laws is a purely mathematical statement in rather complex and abstruse mathematics... It gets more and more abstruse and more and more difficult as we go on." Why is this the case? Feynman offers no explanation, but merely states this as a fact which is certainly a facet of the character of physical law. The interplay between physics and mathematics raises many other questions however: Can mathematics generate new physics (and vice versa)? Is mathematics simply a tool for understanding and predicting phenomena in the physical world, or is it something more profound? The answers to these questions can only be found by looking more closely at the link between mathematics and physics.

3. THE NATURE OF THE RELATIONSHIP BETWEEN PHYSICS AND MATHEMATICS

3.1 Why is Mathematics Useful in the Description of the Physical World?

As Feynman points out at the beginning of his lecture, *The Relation of Mathematics to Physics*, "it is perfectly natural that mathematics will be useful when large numbers are involved in complex situations." [7] He uses the example of a game of chequers in which each possible move is not of itself particularly mathematical in nature, but an analysis of which are the best moves to make becomes a mathematical exercise. This is just an outcome of the nature of mathematics as a formalisation of abstract reasoning. However what we see is that even in the most simple situations (for example two masses attracted by the force of gravitation) the physical laws must be couched in mathematical language.

It is important to distinguish then between these two distinct roles of mathematics in physics, its use in the description of complex phenomena and in the representation of the physical laws. The relative importance of these two applications depends on the extent to which one views physics as a reductionist science. Luke Drury of the Dublin Institute for Advanced Study says "Traditionally physics saw its goal as being the reduction of all phenomena to a few very basic principles… But there is also a school of thought which holds that there are valid areas of physics, which arise from inherent complexity of systems, turbulence for example, and that there are emergent phenomena that you can study in terms of physics, but are not simply reducible to a reductionist paradigm." [8] My own opinion is that a reductionist view of the universe is a particularly elegant one; any theory of the universe in which the behaviour of all phenomena can be reduced to a few simple laws is aesthetically very pleasing. While the application of these laws to real physical situations may produce complex and unexpected results, the important thing is to have a consistent basis from which these results can be derived; the fact that certain phenomena cannot be predicted at present by us is a shortcoming of our theories or our calculational abilities, not of reductionism itself. As Einstein puts it in his lecture *On the Method of Theoretical Physics,* "It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience."

This brings to mind the use of Feynman's 'Babylonian' or 'Greek' approaches to physics; while Feynman advocates a Babylonian approach to physics in its current state, he does concede, "Some day, when physics is complete and we know all the laws, we may be able to start with some axioms, and no doubt someone will figure out a particular way of doing it so that everything else can be deduced." Feynman's Babylonian approach then may be symptomatic of the current incompleteness of physics. The Greek method may only be applicable if the reductionist's viewpoint turns out to be vindicated by a simply expressible Theory of Everything. We will examine the likelihood of this possibility at the end of the essay.

We now turn from the less troublesome explanation of the occurrence of mathematics in describing complex physical situations, to the reason why the fundamental physical laws themselves are mathematical in nature and (as we will see in more depth in section 3.2) why mathematics may be used to deduce new physical laws from the existing ones. The arguments concerning this are varied and complicated and unfortunately only a brief précis of two such arguments can be given here.

The first is propounded by Dennis Dieks of Utrecht University who opines that "mathematics is flexible and versatile and it is the very difference in the nature of mathematics and physics that makes it applicable in the most disparate scientific domains and hence vastly effective."[6] This argument views mathematics as nothing more than an (albeit very useful) tool; it is therefore something of a formalist's view of mathematics[1] and an empiricists view of physics, which while in no way unorthodox, is not the only view that may be held. Luke Drury of the Dublin Institute for advanced studies expresses the perhaps less philosophically discrete view that, "the unreasonable effectiveness of mathematics derives precisely from the fact that it is abstracted physics". His view is that the distinction between Platonists and formalists is in fact ob-

[1] The formalist view of mathematics was championed by David Hilbert who said, "Mathematics is a game played according to certain simple rules with meaningless marks on paper." In the formalist doctrine mathematics is simply a tool for moving from one point to another by a series of logical steps. This is in opposition to the Platonist view in which mathematical truths are eternal and are discoveries by the human mind and not inventions of them.

solete. Mathematics is based on our experience of the world around us but in a sense becomes the ideal things of Plato by the very process of abstraction. However, neither extreme formalism in Hilbert's sense of the word nor extreme Platonism need be invoked to corroborate this explanation.

The most widespread opinion however still appears to be that expressed by Wigner in his paper *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*. "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." [9] This viewpoint is completely justifiable, particularly when one looks at how much of the link between the two subjects is yet to be understood.

3.2 Can mathematics generate new physics (and vice versa)?

The answers to these questions are certainly yes; the generation of new physics by the application of mathematics to existing physical laws forms the third role of mathematics in the physical sciences. The converse generation of new mathematics is also of course being a key part of the relationship between mathematics and physics.

There are innumerable examples of the generation of new physics by mathematics. I have chosen to briefly go through one example explicitly as I found it particularly striking when it was first taught to me. The illustration I have chosen is the Aharonov-Bohm effect. Here we start from Maxwell's equation, $\nabla \cdot \boldsymbol{B} = 0$. The rules of vector calculus tell us that we can write, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ for any vector field $\boldsymbol{A}$, since the divergence of a curl is always 0. We call this field, $\boldsymbol{A}$, the vector potential. This appears to just be a mathematically different way of writing the same thing. However, it is now possible to have a region where $\boldsymbol{B} = 0$ but where $\boldsymbol{A}$ is non-zero. Physically, we might think that an observable change to electrons passing through a region could only be caused by a non-zero magnetic (or electric) field and that $\boldsymbol{A}$, as a mathematical construct could not affect any experimental results. Nonetheless, $\boldsymbol{A}$ does produce a quantum mechanical phase shift in the wave function of electrons, which leads to measurable interference effects. I certainly found this a very surprising result; the fact that we may start from some physical law, apply a set of mathematical steps which do not necessarily follow a prescribed path laid out by physical intuition and arrive at a valid result with real, measurable physical meaning.

Despite also applying to the representation of the most elementary physical laws, this ability to use mathematics to create (or discover?) new physics is what Wigner is really referring to in the title of his paper. Dirac also feels that it is an 'unreasonable usefulness' and expresses this in the introduction to his speech to the Royal Society of Edinburgh in 1939, "The physicist, in his study of natural phenomena, has two methods of making progress: (1) the method of experiment and observation, and (2) the method of mathematical reasoning. The former is just the collection of selected data; the latter enables one to infer results about experiments that have not been performed. There is no logical reason why the second method should be possible at all, but one has found in practice that it does work and meets with remarkable success."[10] Dirac himself famously made extensive use of the happy accident of maths' effectiveness by the prediction of antimatter from his seminal equation. In fact, his advocation of the fruitfulness of such an approach has led to the concept of, "Dirac's methodological revolution according to which the contemporary physical theory should be constructed by working with pure mathematics instead of reflecting conjecturally only on physical phenomena" [6]. Other examples of the successful use of such an approach include the Lorentz transformations, gravitational lensing, the Lamb shift and many, many more [11].

The cases of physics generating new mathematics are also many and well documented. Throughout history the disciplines of mathematics and physics have gone hand in hand and many physicists have also been highly gifted mathematicians. The most obvious example to give might be Isaac Newton whose invention of calculus, called by Newton his *Method of Fluxions*, was formulated in terms of motions and velocities i.e. in physical terms [12].

Other examples of physicists developing new mathematical techniques abound, but a more interesting facet of the interplay is the appearance of extremely recondite mathematics into physical theory and the similar (although generally more recent) phenomenon of physical theory generating new mathematics in unexpected and seemingly inexplicable ways.

We will start first with some examples of esoteric mathematics cropping up unexpectedly in unlikely physical situations. Perhaps some of the most abstract mathematics we can think of is found in the rarefied air of number theory. One of the most famous problems in number theory is the Riemann hypothesis which is concerned with the distribution of zeros of the Riemann zeta function. The Riemann hypothesis is intimately connected with the distribution of prime numbers. However, the distribution of zeros of the zeta function turns out to correspond to physical phenomena such as the spacing of energy levels of heavy atoms [13]. In fact number theory seems to crop up in many seemingly disparate areas of physics such as statistical mechanics, quantum chromodynamics and many more [14].

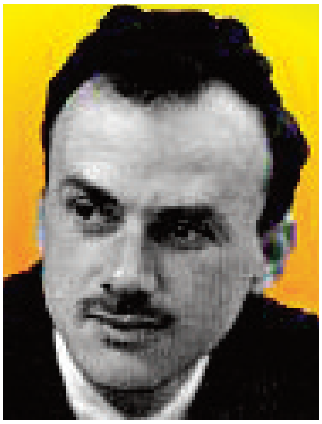
Perhaps even more surprising are physical theories creating new and abstruse mathematics. For example Brendan Goldsmith gives as an example, "quantum field theory has had a significant influence in many areas of geometry from elliptic genera to knot theory". We have examined some of the reasons why mathematics might be able to produce new physics, and even these are questionable in their adequacy. The reasons why physics should be able to create new, seemingly abstract mathematics are even more elusive.

The answer to our question then is a resounding yes, but what has, I hope, been highlighted by this section is the fact that the link between maths and physics is not yet fully understood. The relationship is of such depth that we are currently at the stage of getting fascinating glimpses at the nature and topology of the connections, but are still far from being able to map them in their entirety.

4. POSSIBILITIES FOR THE FUTURE

Here I will present two speculative conjectures on the nature of the link between mathematics and physics and what this may mean for the physical sciences.

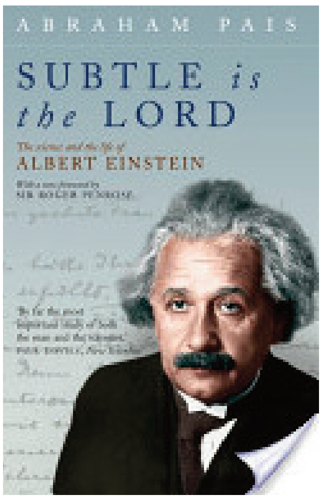
The first is an idea of Paul Dirac's given in his 1939 Edinburgh Royal Society speech, which was entitled 'The Relation between Mathematics and Physics'. After an insightful discussion on the subject he comes to the conclusion that the current state of affairs is unsatisfactory due to "the limitation in the extent to which mathematical theory applies to a description of the physical universe." This is a view which I share. Currently maths and physics both seem to me a little bit like large (perhaps infinite) archipelagos; scientists and mathematicians both attempt to enlarge the islands, and in so doing sometimes find they have made it all the way to another island, or they meet a fellow scientist/mathematician coming the other way.[2] What we may find is that all the islands and promontories turn out to be part of a single landmass and possibly that maths and physics are in fact the same landmass but viewed in different ways. There are so many examples where mathematical ideas have seemed irreconcilably far removed from physical reality and then turned out to be highly relevant or even indispensable to a physical theory, that one may be led to wonder if it is only our incomplete knowledge of physics (and mathematics) which stops us establishing a 'one-to-one' correspondence. An example of the aforementioned: before the advent of quantum mechanics it may have seemed laughable to suggest that observables would be obtained by the action of self-adjoint operators on vectors in an infinite dimensional Hilbert space. When the concept of complex numbers was first introduced, it too may have seemed inapplicable to the physical world. As Wigner points out "Surely to the unpreoccupied mind, complex numbers are far from natural or simple and they cannot be suggested by physical observations. Furthermore, the use of complex numbers is in this



[2] An example from physics might be finding that the electric and magnetic fields are actually two aspects of the same phenomenon, but viewed in different inertial frames by applying the transformations of special relativity. In mathematics a famous example would be Wiles' proof of Fermat's last theorem. This in effect turned out to be a proof of the (ostensibly unrelated) Taniyama-Shimura conjecture which itself was a proof of the correspondence between elliptic curves and modular forms, two seemingly disparate groups of mathematical objects.

case not a calculational trick of applied mathematics but comes close to being a necessity in the formulation of the laws of quantum mechanics."

Dirac shares this view hypothesising that, "the whole of the description of the universe has its mathematical counterpart" and further that "a person with a complete knowledge of mathematics could deduce, not only astronomical data, but also all the historical events that take place in the world, even the most trivial ones." This is an extremely bold conjecture but Dirac goes on to suggest how such a scheme might be realized. He thinks that the state of the entire universe at each instant of time may be characterizable by a single integer: the age of the universe expressed in terms of atomic constants. In terms of these units he gives the age of the universe[3] as approximately 10^{39} a number which he says "characterizes the present in an absolute sense". The apparent unlikelihood of such a correlation is not lost on Dirac but neither does it deter him: "At first sight it would seem that the universe is far too complex for such a correspondence to be possible. But I think this objection cannot be maintained, since a number of the order 10^{39} is *excessively* complicated, just because it is so enormous. We have a brief way of writing it down, but this should not blind us to the fact that it must have excessively complicated properties... There is thus a possibility that the ancient dream of philosophers to connect all Nature with the properties of whole numbers will some day be realised."



These ideas are certainly thought-provoking; it struck me on reading this that a more natural and fundamental unit of time for a clock of the universe might be the Planck time[4]. The age of the universe in these units is approximately 10^{61}, a number of even greater complexity with which to quell concerns. I was also led to think about how connections between the natural numbers and the physical universe might show themselves; perhaps there is a certain range of the natural numbers[5] which all have some particular obscure feature in common which correspond to the inflationary period of the early universe; what particular aspect of the physical universe might correspond to a prime number value of time on the universe's clock? Perhaps at the instant the clock ticks past one of the 62 digit primes of the current epoch the universe displays some measurable feature not observable at other (non-prime) times. I am in some danger now of getting carried away by an idea which, if true, would certainly be of unprecedented profundity, but which also seems absurd to the rational mind. Indeed were the scheme not proposed by such a venerated figure as Dirac, I may not have had the courage to give it any credence. With the hope of doing no disservice to the name of my favourite physicist, a quotation from Einstein seems apt: "I have trouble with Dirac. This balancing on the dizzying path between genius and madness is awful." [15]

The second conjecture is put forward by Stephen Hawking, but before we look at this we must first acquaint ourselves with Gödel's Incompleteness Theorem. This may be the most important result in the history of mathematical logic and roughly states that *For any consistent theory that proves basic arithmetical truths, an arithmetical statement that is true, but not provable in the theory, can be constructed. That is, any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete.*

As Hawking puts it, "Thus mathematics is either inconsistent, or incomplete. The smart money is on incomplete." [16]

---

[3] At the time, the age of the universe was thought to be 2×10^9 years.

[4] This unit is defined only from the fundamental constants $\hbar$, G and c and may be the shortest time interval which can have any physical meaning. The Planck time is given by:

$$t_p = (\hbar G/c^5)^{1/2} \approx 5{,}4 \times 10^{-44} \text{ s}$$

and is the time it would take a photon to travel the Planck length.

[5] Between about 20 million and 2×10^{31}.

He goes on to ask, "What is the relation between Gödel's theorem and whether we can formulate the theory of the universe, in terms of a finite number of principles? One connection is obvious. According to the positivist philosophy of science, a physical theory is a mathematical model. So if there are mathematical results that can not be proved, there are physical problems that can not be predicted."

Throughout this essay we have seen the strength of the link between mathematics and physics. Does Gödel's theorem mean that this link while providing support and progress in physics also means that a Theory of Everything (or at least one based on a finite number of axioms) is impossible? Perhaps. However, despite a slight reluctance to relinquish Feynman's idea of an axiomatic approach being employed "Some day, when physics is complete and we know all the laws", I share Hawking's sentiment about the future.

"Some people will be very disappointed if there is not an ultimate theory that can be formulated as a finite number of principles. I used to belong to that camp, but I have changed my mind. I'm now glad that our search for understanding will never come to an end, and that we will always have the challenge of new discovery. Without it, we would stagnate... I'm sure Dirac would have approved."

REFERENCES

[1] Sir James H. Jeans, *The Mysterious Universe*, Cambridge, CUP, 1948. http://en.wikipedia.org/wiki/History_of_physics

[2] Richard P. Feynman, *The Character of Physical Law*, Penguin Press Science, 1992.

[3] James A. Philip, *Pythagoras and Early Pythagoreanism*, University of Toronto Press, 1966.

[4] Robert Grosseteste and Ludwig Baur, Die Philosophischen Werke des Robert Grosseteste, Munster i.W., Aschendorff, 1912.

[5] Giovanni Boniolo, Paolo Budinich and Majda Trobok (Eds), *The Role of Mathematics in the Physical Sciences*, Springer, Netherlands, 2005.

[6] Richard P. Feynman, *The Relation of Mathematics to Physics,* Part II of Ref. [3].

[7] Panel Discussion, *The Interface between Mathematics and Physics*, www.maths.tcd.ie/pub/ims/bull58/riadiscourse.pdf

[8] Eugene P. Wigner, *Communications in Pure and Applied Mathematics,* Vol. 13, No. 1, John Wiley and Sons, New York, 1960.

[9] Paul A.M. Dirac, *Proceedings of the Royal Society (Edinburgh),* Vol. 59, 1938-39, Part II, pp. 122-129.

[10] For more details on each of these (and for further examples) see Mark Colyan's, *The Indespensability of Mathematics,* Section 4.4 (Oxford University Press, 2001), Wigner's paper.

[11] Florian Cajori, *History of Mathematics* (5th Ed.), pp. 191-205, American Mathematical Society Chelsea Publishing, 1991.

[12] Marcus P.F. du Sautoy, *The Music of the Primes*, Harper Collin Publishers Inc., New York, 2003. For more info see: http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin/zeta/physics.htm

[13] Quoted from Abraham Pais, *Subtle is the Lord: the Science and the Life of Albert Einstein*, Oxford University Press, Great Britain, 2005.

[14] Stephen W. Hawking, *Gödel and the End of Physics*, http://www.damtp.cam.ac.uk/strings02/dirac/hawking/

Calum Templeton
Master of Arts (MA), Cambridge
Physical Natural Sciences