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The effects of chaos on business operations

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I. Resumen y conclusiones

El tema central de esta tesis son los efectos que tiene el caos sobre las operaciones comerciales. Con el fin de explicar este tema en mayor detalle, esta introducción consta de dos secciones que explican las operaciones comerciales y el caos. Para aquellas personas que leen muy rápido, se resume el objetivo de esta tesis y el resultado a continuación:

El punto de partida es el trabajo que realizó Edward Lorenz acomienzos de la década de 1960 en el Instituto de Tecnología de Massachusetts. Por razones obvias, el pronóstico del tiempo es casi tan antiguo como la humanidad. El pronóstico del tiempo contemporáneo funciona de la siguiente manera. La persona tiene que determinar los datos climáticos del día (la temperatura, la presión atmosférica, etc.), emplear las leyes de la termodinámica y la mecánica clásicas y calcular cuál va a ser el clima partiendo de esa información. Desde un punto de vista puramente físico es aburrido porque las leyes de la termodinámica y la mecánica clásicas necesarias tienen siglos de antigüedad, y hasta este momento nadie duda de su validez en esta área; es más, no hay motivos para hacerlo. Calcular el clima del día siguiente, sin embargo es *sumamente* tedioso. Incluso al día de hoy existen muy pocas herramientas informáticas disponibles y se sabe que los parámetros de entrada (el clima de hoy) son demasiado imprecisos. Por lo tanto, el pronóstico del tiempo a más largo plazo fue posible cuando aparecieron los ordenadores. Está claro que parámetros de entrada diferentes conducen a pronósticos diferentes, pero Edward Lorenz hizo una observación curiosa siendo pionero en el área: cambiar poco los parámetros de entrada casi no influenciaba el pronóstico del tiempo en el corto plazo, como todo el mundo esperaría, pero en el largo plazo, lo cambiaba radicalmente, incluso la modificación más sutil podía producir una diferencia en si lloviera en 30 o 31 días. Hasta el movimiento aéreo de una mariposa podía cambiarlo. Así nació la expresión "efecto mariposa". Por el contrario, los promedios espaciales y temporales son sumamente constantes. En los últimos 50 años, la temperatura promedio de la superficie de la Tierra aumentó una fracción de un punto porcentual, y estamos (correctamente) preocupados al respecto. Sin el efecto invernadero, el cambio sería mucho menor o la temperatura promedio sería mucho más constante.

Ambos efectos, 1) la imposibilidad de realizar un pronóstico del tiempo a largo plazo y 2) la constancia de las cantidades promedio, se entienden perfectamente. El efecto mariposa es la consecuencia del caos (matemático). La constancia de las cantidades promediadas se relaciona con las leyes de conservación, por ejemplo, la conservación de la energía. Se puede encontrar una introducción a este tema en la segunda sección del prólogo y un estudio detallado en el Capítulo II.

Sin embargo, esta tesis no trata ni de meteorología ni de física, trata de negocios y economía. Se podría decir que una tarea importante de la función de un gerente o un economista es la del realizar "pronósticos y planificar", y dicha tarea está estrechamente vinculada al pronóstico del tiempo. No tanto por la técnica o las ecuaciones que se emplean, sino por el proceso metodológico. Se han de tomar parámetros de entrada, como las cifras de las ventas, los costes, etc. del día, para después "calcular" cifras de futuro relevantes, como los ingresos del próximo año, por ejemplo. Las comillas en la palabra "calcular" se deben a que no son ecuaciones rigurosas como las de la meteorología, de todos modos, existen ecuaciones o pueden ajustarse por medio de modelos de regresión en los que se da pie a ciertos errores de "pronóstico". Con bastante frecuencia, se programan medianteun software informático, por lo tanto, existe al menos una posibilidad de que ocurran cosas como el efecto mariposa en los negocios y la economía. Y sin duda, ocurren. Entonces, existe un umbral (natural) que indica cuán largo o detallado puede ser un plan o un pronóstico. Reconocer dicho umbral es esencial y también es el objetivo fundamental de esta tesis. Al encontrarse con una situación caótica en los negocios o en la economía, se debe buscar la manera de describirla; una posibilidad, como lo demuestra esta tesis, es el uso de las cantidades conservadas.

El resto de la tesis se organiza de la siguiente manera. El Capítulo II contiene la parte teórica de la tesis; resume los efectos del caos matemático y las leyes de conservación lo necesario para poder comprenderla. En la primera afirmación, en la página 27, se resume el resultado principal: **las cantidades conservadas no pueden exhibir un comportamiento caótico**. También aporta la relación con los negocios con un segundo resultado importante, como se explica en la segunda afirmación, en la página 34: **existe una cantidad conservada en los negocios y la economía**. Para probar esto se han empleado los análisis realizados por Sato y Ramchandran (2014)[1]. El Capítulo III también puede considerarse una parte importante de la tesis ya que contiene, y resuelve, un caso específico en el que se aplican los resultados teóricos. Se analizan cuatro situaciones comerciales diferentes para estudiar los efectos del caos y las cantidades conservadas subyacentes. Se demuestra que la ubicación óptima de un depósito puede presentar efectos caóticos, por lo tanto, (en general) es inútil determinarla. Por el contrario, el costo del depósito es una cantidad conservada y no muestra efectos caóticos. En la Sección 2 del Capítulo III, se aplica un modelo de curva de aprendizaje en dos partes para pronosticar las cifras de venta de un producto nuevo. Puede demostrarse que el desarrollo (oportuno) de la participación en el mercado no puede predecirse porque tiene un comportamiento caótico, a diferencia de la participación final en el mercado, que no presenta caos porque es una cantidad conservada. En la Sección 3 del Capítulo III, se analiza el modelo de difusión de la comercialización (es un modelo estándar para calcular el desarrollo en el mercado de un producto nuevo). Se publicó en base a una evidencia de la presencia de caos en el modelo de difusión ya en 1993[2]. Aquí se demuestra que los efectos caóticos aparentes se deben a una incomprensión matemática. La corrección de estos errores hace que desaparezcan los efectos caóticos y simplifica el uso del modelo de difusión. En la cuarta sección del Capítulo III se demuestra que un mercado financiero típico nunca es estable y por lo tanto presenta un comportamiento caótico. La tercera afirmación relevante se recoge en la

[1] Sato, R. and Ramchandran, R. (2014).-"Conservation Laws and Symmetry: Applications to Economics and Finance", Springer

[2] Weiber, R. (1993).- "Chaos: Das Ende der klassischen Diffusionsmodellierung", Marketing ZFP, 1, 35-46.

página 66 y la constituye **el motivo por el que la especulación nunca puede conducir a la ganancia en el largo plazo**.

1. La situación inicial en las operaciones

Como se mencionó anteriormente, las predicciones cuantitativas son esenciales en los negocios y la economía, de ellas derivan los planes para el futuro y a menudo son las entradas fundamentales que requieren una estrategia a largo plazo. Los resultados de las ciencias de la gestión y, más específicamente, el uso de ordenadores facilitan la posibilidad de hacer predicciones cuantitativas. Hasta aquí es absolutamente análogo al pronóstico del tiempo. Ya sea en la física, los negocios o la economía, se supone que el mundo es determinista. Cabe destacar que si algo no es determinista, ocurrirá de casualidad. Si algo es determinista, las personas tratan de averiguar qué causas tienen qué efectos. En la física, existen dos leyes elementales, son básicamente conocidas, como sería el caso de la ecuación de Newton en combinación con la de Maxwell'. En los negocios y la economía, la humanidad está apartada. En parte, existen algunos indicios que las explican y, en parte, mayormente hay reglas empíricas. Por lo tanto, si un modelo de negocios o económico no conduce a un resultado apropiado, existen dos motivos posibles:

- Los parámetros de entrada son incorrectos o imprecisos.
- El método de predicción no es bueno o es menos preciso en el largo o corto plazo.

En cuanto a sus respectivos efectos, ambos motivos son indistinguibles. Esto puede considerarse un grave problema, pero hace hincapié en el hecho de que es muy importante considerar el rango de validez de todo resultado. Todo estudiante de Física aprende en su primer año que cualquier cantidad medida tiene un margen de error. Si, como mínimo, no se tiene esto en cuenta, la cantidad medida es completamente inútil. Cuando se incluye este margen de error, cualquier modelo de física obtiene su rango de validez, puede considerarse evidente, y de hecho lo es. Lo sorprendente es que rara vez se utiliza fuera del campo de la física. Ni siquiera en la economía es fácil encontrarlo. A veces aparecen predicciones como "el PBI crecerá entre el 2,6 % y el 3,2 % el próximo año", pero ni siquiera ese suele ser un margen de error real, en general es el diferencial del pronóstico realizado en forma independiente por distintas instituciones. En especial en los negocios a veces existe una mala interpretación de las estadísticas. Se escuchan afirmaciones del tipo "esta estrategia es correcta en un 80 %", lo que significa que es bastante apropiada en el 80 % de los casos, pero en el 20 % de los casos puede ser un disparate absoluto. Dicha interpretación contiene dos errores fundamentales. En primer lugar, no existen 100 situaciones comerciales idénticas y 80 están yendo por un camino y 20 por otro. En segundo lugar, si un modelo o procedimiento a veces está equivocado, en general está equivocado. ¡Con un contraejemplo es suficiente para falsear una teoría!

A partir de esta información, debe quedar claro que rara vez se observan los efectos caóticos en los negocios y la economía. Si algo tiene un resultado completamente diferente, aunque la configuración inicial fuera casi idéntica, se podríaafirmar que es una situación en la que el modelo no es correcto.

2. El efecto del caos matemático en los modelos de negocios

El caos se da cuando las causas pequeñas tienen efectos grandes. Hace más de cien años, los matemáticos encontraban funciones como f(x) presentada en la ecuación [3] de la página 17. En general, si el argumento de una función (aquí x) varía poco, f(x) (el resultado) también varía muy levemente. Pero en la función mencionada anteriormente tenemos una situación en la que $|f(x) - f(x + \varepsilon)|$ siempre estará en promedio ½ completamente independiente de ε. Esto es particularmente sorprendente para ε muy pequeños. Si se supone un parámetro de entrada a x y un parámetro de salida a f(x), desde un punto de vista teórico, el resultado es estrictamente determinista. Sin embargo, como se mencionó en la sección anterior, toda cantidad medida tiene un margen de error. Incluso en el caso de que fuera arbitrariamente pequeño, el resultado siempre varía ± ½. Es decir, en todas las situaciones prácticas el resultado solo puede determinarse con esta precisión (que es bastante grande porque f(x) permanece entre 0 y 1 en este caso. En otras palabras, tenemos un margen de error de ± 50 %.) Por lo tanto, si algo está regido por dicha función, no puede haber predicciones precisas. Como voy a demostrar más adelante, la antes mencionada f(x) describe la cantidad de dinero de una cuenta bancaria (manejada de forma extraña) (para conocer más detalles, ver la subsección 3.1 del Capítulo II). Esta es la evidencia más antigua de la presencia de caos en una situación comercial (en este caso, en contabilidad). Sin embargo, ha sido señalada por un matemático, Peitgen y Richter (1984)[3], no por un economista ni un profesional de las ciencias de la gestión.

Cabe destacar que dichos efectos pueden aparecer fuera del mundo de las fórmulas y las matemáticas. A modo de ejemplo, consideraremos un congreso anual. La fecha límite para presentar los trabajos es el 30 de junio a las 12 horas (en punto). Exactamente a las 12 horas del 30 de junio nos encontramos con un comportamiento muy caótico. El trabajo aparecerá un año más tarde o antes dependiendo de si se presentó una milésima de segundo antes o después. En las situaciones comerciales, la planificación se hace aplicando algunas fórmulas matemáticas y algunas reglas, como por ejemplo, la fecha límites es... Por lo que es sumamente probable que se desarrolle el caos. Por el contrario, Peitgen y Richter (1984)[3] fue el primero en someter a debate un ejemplo de contabilidad. Quizás porque es bastante artificial, nunca triunfó en el mundo de los negocios. Más tarde, Weiber (1993)[4] expuso que un modelo estándar de investigación de mercado cuantitativa (en este caso, el modelo de difusión) puede presentar caos y es por lo tanto de uso limitado. Dichas noticias deberían haber sido un problema muy serio para el modelo de difusión, pero sorpresivamente no lo fueron. Todavía hoy se sigue usando el mismo modo. Es más sorprendente aún que los efectos caóticos de Weiber (1993)[4] desaparecen por completo si se usa

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[3] Peitgen, H.O.; Richter, P. H. (1984).- “Harmonie in Chaos und Kosmos”, “Morphologie komplexer Grenzen; Bilderaus der Theorie dynamischer Systeme”, Universität Bremen, Forschungsschwerpunkt dynamische Systeme

[4] Weiber, R. (1993).- “Chaos: Das Ende der klassischen Diffusionsmodellierung”, Marketing ZFP, 1, 35-46.

la matemática correcta en el modelo de difusión. Esto se demuestra en la Sección 3 del Capítulo III de esta tesis. (Si se usa la matemática correcta es mucho más fácil aplicar el modelo de difusión, pero al día de hoy se sigue usando una matemática incorrecta, lo que como máximo da una aproximación.)

Además de los enfoques sobre el caos en los negocios del último párrafo, fue Grabinski (2004)[5] quien notó las consecuencias del caos matemático en los negocios y la economía. Más tarde, Grabinski (2007[6], 2008[7]) ofreció algunos indicios más detallados y demostró el efecto del caos en el almacén óptimo si hay solamente dos clientes. Este fue el punto de partida de esta tesis, en la Sección 1 del Capítulo II, lo extendí hasta tres clientes y más.

Si hay presencia de caos, es inútil seguir calculando valores para cantidades que varían caóticamente. Appel y Grabinski (2011)[8], además de Appel, Dziergwa y Grabinski (2012)[9] utilizaron cantidades conservadas para describir situaciones caóticas en mercados financieros, este fue mi punto de partida para demostrar las leyes de conservación en los negocios y la economía (ver subsección 3.2 del Capítulo II). En situaciones comerciales en las se encuentra caos a lo largo de la investigación, se busca, asimismo, cantidades conservadas y se demuestra de forma explícita que no varían caóticamente.

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[5] Grabinski, M. (2004).- “Is there ‘chaos’ in management or just chaotic management?”, Complex Systems, Intelligence and Modern Technology Applications, Paris.
[6] Grabinski, M. (2007).- “Management methods and tools: Practical know-how for students, managers, and consultants”, GablerVerlag, Wiesbaden.
[7] Grabinski, M. (2008).- “Chaos – limitation or even end of supply chain management”, High Speed Flow of Material, Information and Capital, Istanbul.
[8]Appel, D.; Grabinski, M. (2011).- “The origin of financial crisis: A wrong definition of value”, PJQM Vol. 3.
[9]Appel, D.;Dziergwa, K.; Grabinski, M. (2012).- “Momentum and reversal: An alternative explanation by non-conserved quantities”,Int. Jour.Of Latest Trends in Fin. & Eco. Sc., 2 1, p. 8.

3. Conclusiones y otras investigaciones

A modo de conclusión, existen muchas más áreas en las que se deberían analizar los efectos caóticos. En este caso, haré hincapié en las implicaciones de las investigaciones aplicadas realizadas por compañías, bancos, consultoras, etc. Aunque es posible que existan otras áreas de la economía en la que se produzcan este tipo de situaciones, pretendo poner el foco solamente en dos: los efectos sobre el software (desarrollo y uso) y la repercusión en las estrategias de las empresas.

3.1 Los efectos sobre el software

Esta tesis presenta los efectos caóticos en las operaciones logísticas, como la planificación de depósito y la planificación tal como se realiza en comercialización. Los cálculos se realizan, primero, derivando primero el algoritmo y, después, de forma individual. En las situaciones comerciales, todo esto se efectúa con un software adquirido en el mercado. Los sistemas de planificación de los recursos de la empresa (ERP, por sus siglas en inglés), como SAP, se utilizan para las operaciones elementales y cotidianas, pero para la ubicación óptima de almacenes también existen programas prefabricados, por lo tanto, los desarrolladores y los usuarios de software deben ser conscientes de los posibles efectos caóticos.

En la parte principal de mi tesis, menciono el tiempo de cálculo prolongado (tiempo de CPU) para los cómputos en presencia de caos. En la figura 9, por ejemplo, se utilizaron unas 100 horas de tiempo de CPU en una potente estación de trabajo. Esto se debe de forma fundamental a la alta precisión que se requiere en un régimen caótico. Por lo tanto, un programa estándar (ya sea Excel o SAP) nunca obtendrá resultados verdaderos en presencia de caos. A primera vista, no parece un gran obstáculo. En un régimen caótico, todo resultado es completamente inútil, por lo tanto, no es necesario conocerlo con exactitud. Parece ser aleatorio. Bien sea que estos "números aleatorios" sean producidos por cálculos sumamente precisos o en parte por el efecto de un software impreciso podría ser algo de interés puramente académico. Sin embargo, el software estándar produce resultados finales, como el tiempo de amortización,y al usuario solo le interesa este número final. Con el fin de calcular este número, el software en uso realiza varios pasos (en general, son muy diversos; de lo contrario, no sería necesario usar un software). Tal vez algunos de los números que se producen en los procesos entre medias muestren un comportamiento caótico y,por lo tanto, el resultado (sea o no caótico) estará probablemente equivocado.

Resumen y conclusiones

Con el fin de analizar este punto con mayor claridad, tomo una cuenta bancaria (caótica) manejada de manera extraña. Para conocer más detalles, ver la subsección 3.1 del Capítulo II. Allí, la figura 12 de la página 30 muestra el balance de la cuenta después de cada período. Parece ser un número completamente aleatorio. Calcular este número aleatorio es un desafío numérico. Si se hubiera realizado el cálculo con, por ejemplo, Excel, la imagen se vería igual de aleatoria, pero los detalles serían completamente diferentes (se aprecia un efecto similar en la figura 7 de la página 21, aunque este es puramente matemático). Después de 120 períodos, hay, por ejemplo, 817.959,10 euros en la cuenta en realidad, mientras que un cálculo de Excel reporta 580.250,92 euros. Por supuesto que a nadie le interesan los detalles de la figura 12. Sin embargo, como se argumenta en la subsección 3.1 (del Capítulo II), el dinero total retirado de la cuenta (más el interés) es una cantidad conservada, por lo tanto, no debe fluctuar caóticamente, pero de hecho lo hace, lo que se muestra en 13 de la página 31. Es básicamente la suma de los puntos caóticos de la figura 12. El cálculo se realizó con una gran precisión. Si no se hubiera empleado tanta precisión, la figura 13 también podría mostrar un comportamiento caótico.

En síntesis, existen algunas lecciones importantes para los desarrolladores de software:

- En cada paso dentro del algoritmo, debe quedar claro si es posible que existan efectos caóticos o no.
- El software debe probarse para comprobar la existencia de efectos caóticos en el resultado final y en todos los resultados internos.
- Lo mejor es usar cantidades conservadas solo cuando sea posible.

Es muy importante que los usuarios del software sean conscientes del caos. Como mencioné antes, usar el programa Excel para calcular el balance de cuenta de la subsección 3.1 (del Capítulo II) es muy simple pero el resultado eserróneo. En todo tipo de planificación o predicción, siempre se debe suponer un margen de error en los datos introducidos. Se puede suponer una distribución de Gauss con un ancho normal. (Esto es completamente análogo al procedimiento de la subsección 2.2 del Capítulo III.) Es necesario tener en cuenta que la distribución de datos no suele comportarse como la distribución de Gauss y que puede depender de la situación en particular. Sin embargo, este comentario, por lo demás importante, no tiene relevancia en este caso. Siempre se puede escoger una distribución de Gauss (o cualquier otra que sea conveniente). Variar los datos introducidos de este modo conducirá a una variación en los datos de salida. Si el resultado varía con la misma distribución, y su ancho adquiere el mismo orden de magnitud, no hay caos. De todos modos, es necesario realizar dicho análisis, incluso si el caos queda excluido porque, por ejemplo, se utilizan cantidades conservadas. El análisis produce el margen de error en los datos de salida. Y sin el margen de error, cualquier resultado es inservible.

Si dicho análisis produce efectos caóticos (cuando la distribución de salida es aleatoria), no se debe tomar en cuenta ningún resultado. Existe la posibilidad de que el algoritmo del software sea únicamente caótico. Se puede probar con software con un algoritmo completamente alterado. Sin embargo, lo más probable es que dicho software no esté disponible y se deba considerar que el caos domina en realidad. Por lo tanto, en este caso no es posible realizar una planificación y el futuro es tan impredecible como el clima en el largo plazo. Usar esos datos aun así es imprudente. En este caso, nada es mejor que los datos erróneos. Hay que tener en cuenta que el uso de datos caóticos ocurre con bastante frecuencia. Para peor, algunas empresas pagan para obtener datos caóticos. En Gran Bretaña, las industrias que dependen del clima, como las del helado, pagan a los departamentos de meteorología para obtener un pronóstico del tiempo en el largo plazo. Los departamentos están encantados con tomar el dinero y entregar algo que es tan útil como arrojar los dados.

Hay que tener en cuenta que para detectar el caos con rigor se necesita mucha potencia de cálculo, la que no suele estar disponible para el usuario del software. Además, en la mayoría de los programas como Excel, no hay posibilidades de aumentar la precisión arbitrariamente. Sin embargo, a veces es posible reducir la precisión. Por ejemplo, si el software tiene una precisión de dieciséis dígitos, se puede reducir a ocho dígitos. Todos los resultados (caóticos o no) permanecerán idénticos dentro de los primeros dígitos, de lo contrario, algo está mal. Si son idénticos, no es prueba de que exista la precisión suficiente, pero es un indicio de que lo más probable es que esté bien.

Para resumir, estas son algunas recomendaciones para los usuarios de software:

- Tratar de usar y considerar únicamente cantidades conservadas.
- Variar siempre los datos introducidos y ver qué sucede con los datos de salida. Además del caos, se deriva el margen de error.
- Realizar todos los cálculos dos veces: con la precisión estándar y con la mitad de la precisión estándar. Se deberían obtener resultados al menos muy similares.

Para cerrar esta subsección, se recogen algunos comentarios para los usuarios de software y los desarrolladores. Variar los datos introducidos dentro de una distribución de Gauss es sencillo pero puede ser sumamente tedioso. La posibilidad de "hacerlo a mano" queda excluida en la mayoría de los contextos comerciales. Por lo tanto, se debería contar con un software adicional que produzca una distribución de Gauss a partir de una cadena de datos en el formato particular. Según tengo entendido, dicho software no está disponible en el mercado. Por lo tanto, habría que producirlo o encargárselo a un fabricante de software especial. Para otros desarrollos de software estándar, es muy recomendable tener un revisor automático de caos. Algunos programas (como Microsoft Project) cuentan con extensiones pero no todos. Producir esas extensiones estaría dentro de las posibilidades de negocio actuales.

3.2 Las repercusiones en las estrategias de las empresas

Esta tesis trata principalmente sobre las operaciones más que sobre las estrategias. Sin embargo, las repercusiones más importantes pueden encontrarse en las estrategias. Básicamente, demuestro que muchas formas de planificación en el largo plazo son imposibles debido al caos, lo que es análogo a la imposibilidad de obtener un pronóstico del tiempo en el largo plazo. Pero la estrategia es un plan o una manera de alcanzar un objetivo en el largo (o mediano) plazo. Y este objetivo es el resultado de una planificación en el largo plazo. Por lo tanto, las personas hablan de "planificación estratégica” cuando se refieren a la planificación en el largo plazo (en general un plan para un par de años en lugar de un plan mensual).

De este modo, la mayoría de las estrategias puede verse afectada por el caos, excepto lasestrategias triviales. Con estrategias triviales, me refiero a situaciones en las que el resultado es evidente de antemano. Por ejemplo, la industria militar suiza tiene un solo cliente: el Ministerio de Defensa. Según se establece en la constitución, el Ministerio de Defensa tiene que comprar productos suizos. En general, producen sistemas de armas avanzados con autorización, por lo que no sorprende que la industria militar suiza pueda planificar sus ingresos con una precisión de ±1 % a lo largo de cinco años. No hay efectos caóticos, pero la estrategia de planificación puede considerarse una farsa de todos modos.

Para ver los efectos caóticos en estrategias "comunes", presentaré un ejemplo. Con el fin de encontrar una estrategia apropiada, las empresas planifican, por ejemplo, los ingresos de A, B, y C durante los próximos dos años. "A" pueden ser los ingresos propios y "B" y "C" los ingresos de dos competidores importantes. (Alternativamente, A, B y C pueden representar los ingresos de tres productos diferentes.) Existen seis posibilidades:

1. A > B > C — uno mismo es el mayor, y B es mayor que C
2. A > C > B — uno mismo es el mayor, y C es mayor que B
3. B > A > C — B es el mayor, y uno mismo está en el medio
4. C > A > B — C es el mayor, y uno mismo está en el medio
5. B > C > A — B es el mayor, y uno mismo es el menor
6. C > B > A — C es el mayor, y uno mismo es el menor

Dependiendo de cuál de los seis escenarios sea el más probable en el futuro, se necesitan hasta seis estrategias diferentes. En el caso de cinco o seis, probablemente sea mejor ser líder en calidad, mientras que en el caso de uno o dos, ser líder en precio no está mal. Por lo tanto, uno puede tratar de planificar o estimar los ingresos de A, B y C para los próximos diez años a fin de escoger una estrategia adecuada. Esta es la tarea habitual de un gerente que trata de encontrar una estrategia conveniente. (Sin embargo, los modelos reales de un ajuste estratégico suelen ser más avanzados e involucran más cantidades que planificar,pero con esta simplificación alcanza para ver el punto.)

Fue Grabinski (2007)[6] quien notó que *cada* cantidad tiene un margen de error y lo mismo ocurre con una cantidad planificada, como los ingresos. Usando un modelo de una ecuación diferencial simple, Grabinski (2007)[6] descubrió que la incertidumbre (= margen de error) en una planificación aumenta exponencialmente a lo largo del tiempo. En otras palabras, la incertidumbre de una planificación de dos años no es el doble de alta que la de un año sino que crece exponencialmente. Matemáticamente, la incertidumbre u (medida, por ejemplo $\pm$ porcentaje de error) aumenta de la siguiente manera:

$$u = u_0 \cdot \left(e^{t/\tau} - 1\right) \qquad [\,1\,]$$

Aquí t es el tiempo. (t = 2 años en el caso de un período de planificación de dos años) τ es una constante (dimensión temporal) que describe un período de cambio normal. En el campo de la Internet los cambios son rápidos, por lo tanto, τ es pequeña en este caso, en general, menor que un año. En el caso de la industria militar suiza, los cambios en el mercado son muy lentos, por lo tanto, τ es muy grande (muchos años). El prefactor u_0 determina la fuerza de la incertidumbre. Mientras que τ es determinada por el entorno (mercado), u_0 depende de la destreza del planificador o de la calidad del procedimiento de planificación. Cabe destacar que dicho aumento de la incertidumbre también es un efecto caótico. En un mundo *idéntico*, todos los resultados (p.ej., el desarrollo de los ingresos) son idénticos. Sin embargo, existen muchos parámetros de entrada, algunos no se conocen e incluso los que se conocen no se conocen con exactitud, varían en una cierta (pequeña) cantidad, y esta variación causa un efecto significativo a lo largo del tiempo. Por lo tanto, no sorprende que la incertidumbre aumente exponencialmente en el tiempo en [1], cf. [7] o la figura 8.

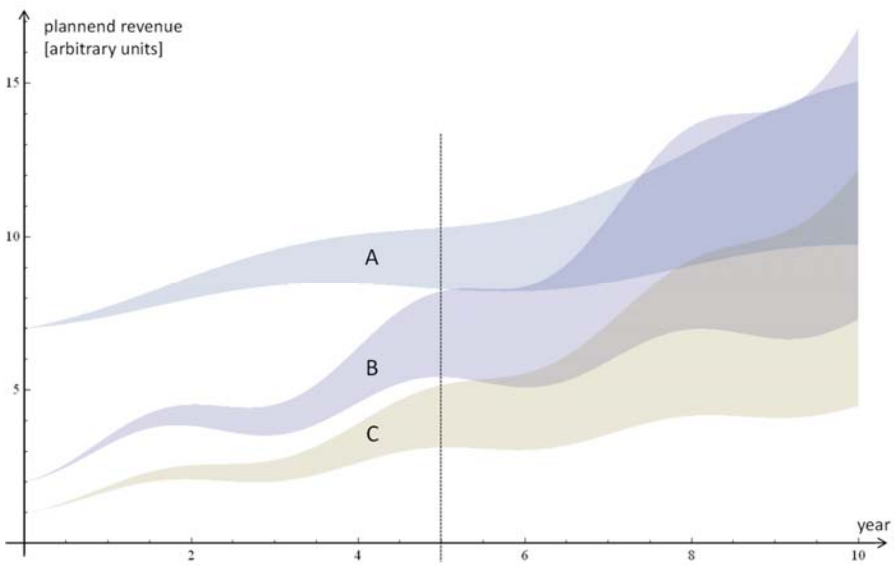

Figura 1: Planificación de ingresos para A, B y C, con el margen de error incluido

Resumen y conclusiones

Desde luego que todo esto también es cierto para la planificación de los ingresos futuros. Los ingresos no están dados por una línea (curva). La línea más bien aumenta exponencialmente en ancho como se muestra en la figura 1. Hasta t = 5 años, se puede estar seguro de que se puede aplicar A > B > C, luego, empiezan a aparecer los efectos caóticos. En el caso de t = 10 años no está nada claro qué ingresos serán mayores o menores. En este ejemplo, se podría hacer una estrategia a cinco años, pero sería imposible escoger una estrategia a diez años. Dependiendo de la situación, es un problema más o menos grave. En especial si la estrategia exige decisiones en el largo plazo, hay que ser muy cauteloso. Es habitual que las distintas estrategias involucren inversiones inmobiliarias, que operan a una escala temporal de unos 20 años. En el caso de la figura 1, sería imposible escoger una estrategia en esas circunstancias, es mejor arrojar los dados.

Si las estrategias en el largo plazo derivan de parámetros cuantitativos como en el llamado ajuste estratégico, siempre habría que hacer lo siguiente:

- Encontrar el margen de incertidumbre para todos los parámetros de entrada relevantes.
- Determinar un horizonte oportuno en el que no importe la incertidumbre (cinco años en la figura 1).
- Comparar este tiempo con el tiempo en que una estrategia debe ser válida (p.ej., el tiempo de amortización) y decidir si la estrategia tiene sentido o no.

Una fusión o absorción tal vez sea una parte de una estrategia. Como una absorción es en términos generales una inversión, todo lo anterior es verdad e importante. Sin embargo, las fusiones y absorciones presentan algunos problemas adicionales debido al caos y en especial a los resultados de esta tesis. Se puede ver fácilmente si se considera la figura 28, que muestra el valor del mercado y el valor conservado (real) respaldado por medio del futuro flujo de caja y su precio de mercado. En primer lugar, una empresa normal que cotiza en la Bolsa está considerablemente sobrevalorada en comparación con el flujo de caja que finalmente produce. En segundo lugar, su precio de mercado varía caóticamente, por lo que es imposible escoger el momento correcto para comprarla (del mismo modo que lo es encontrar el momento correcto para colocar el dinero en el "23" de la ruleta).

Desde esta perspectiva, las fusiones y las absorciones funcionan como hacer apuestas. Esta podría ser en parte la explicación de por qué fracasan con más frecuencia (alrededor de 2/3 de los casos) que triunfan (1/3 de los casos). Hay que tener en cuenta que ambos efectos desaparecen si una empresa paga por la otra otorgando una cierta cantidad de acciones de la nueva empresa (fusionada). Sin embargo, aunque con frecuencia hay un intercambio de mercancías en las absorciones, al menos una parte se paga en efectivo. Es más, una empresa que cuenta con mucho efectivo debería usarlo. Y también existen las empresas privadas grandes que desean absorber a una empresa pública. Considerando mis argumentos anteriores, son pocas las posibilidades que tienen de lograr una adquisición exitosa. A modo de ejemplo, se puede tomar a German Schaeffler KG, propiedad de Maria-Elisabeth Schaeffler y su hijo Georg F. W. Schaeffler en 2010. Intentaron y consiguieron adquirir Continental AG, un gran fabricante de neumáticos alemán, ydesde luego, pagaron en efectivo por las acciones de Continental. Como era de prever, concluyó en un desastre financiero porque tuvieron que pagar demasiado por cada acción. Básicamente, pagaron 10 mil millones de euros por algo que al año siguiente valía 2 mil millones de euros.

I. Summary and Conclusions

The focus of this thesis is on effects of chaos on business operations. In order to explain this in more details, there are two sections in this introduction explaining business operations and chaos. For the very fast reader the goal of this thesis and the result are as follows:

The starting point is Edward Lorenz working in the early 1960s at the Massachusetts Institute of Technology. Weather forecast is almost as old as mankind for obvious reasons. Modern weather forecast works as follows. One has to determine today's weather data (temperature, air pressure, etc.), use the laws of classical mechanics and thermodynamics, and calculate the future weather from it. From a pure physical point of view it is almost boring, because the necessary laws of classical mechanics and thermodynamics are centuries old. And up to now nobody doubts their validity in this area; furthermore there are no hints for it. Calculating tomorrow's weather is however *extremely* tedious. Even today there is still too little computing power available and the input parameters (today's weather) are known too imprecise. Therefore longer term weather forecast was only possible when computers were available. And Edward Lorenz was a pioneer in this area. It is clear that different input parameters will lead to different forecasts. But Edward Lorenz made a very strange observation: Changing the input parameters very slightly had almost no influence on short term weather forecast as everybody would expect. But the long term forecast changed dramatically. Even the slightest change could produce a difference like whether it will rain in 30 or 31 days. Even the air movement of a butterfly could cause it. From this the expression "butterfly wing effect" has been born. In contrast to this, spatial or timely averages are extremely constant. In the last 50 years the average surface temperature of the earth increased by a fraction of a percentage point, and we are (rightly) worried about it. Without the greenhouse effect the change would be much smaller or the average temperature would be much more constant.

Both effects, the impossibility of long term weather forecast and the constancy of average quantities, are perfectly understood. The butterfly wing effect is the consequence of (mathematical) chaos. The constancy of averaged quantities has to do with conservation laws like for example the conservation of energy. Some introduction to it can be found in the second section of the introduction, and a detailed consideration in chapter II.

However, this thesis is neither about metrology nor about physics. It is about business and economics. One important task of a manager or economist is (arguably) "forecasting and planning." And such a task is very closely related to the weather forecast. Not so much by the technique or equations to be used, but by the principle process. One has to take input parameters like today's sales figures, costs, etc. and then "calculate" relevant future figures like for example next year's revenue. The quotation marks in the word calculate are due to the fact, that there are not so rigorous equations like in metrology. But there are equations or they can be adjusted through regression models. Quite often they are programmed in computer software. So there is at least a chance that things like the butterfly wing effect will happen in business and economics. And indeed it does. So there is a (natural) threshold how long or detailed a plan or forecast can be. Recognizing this threshold is very essential, and it is the main goal of this thesis. Having a chaotic situation in business or economics, one must find a way how to describe it. One possibility, as proved in this thesis, is the use of conserved quantities.

The rest of the thesis is organized as follows. Chapter II is constitutes the theoretical part of the thesis. It summarizes the effects of mathematical chaos and conservation laws as far it is necessary to understand this thesis. In statement 1 on page 27 the main result is summarized: Conserved quantities cannot show chaotic behavior. It also brings the connection to business with a second important result as stated in statement 2 on page 34: There is one conserved quantity in business and economics. Chapter III can also be considered as substantive part of the thesis since it includes and solves a specific case where the theoretical results are applied. Four different business situations are scrutinized for chaos effects and the underlying conserved quantities. It is shown that the optimal number of warehouses can show chaos effects. Therefore it is (in general) useless to determine it. In contrast warehouse cost is a conserved quantity and does not show effects of chaos. In section 2 of chapter III a newly developed two party learning curve model is applied to forecast the sales figures of a new product. It can be shown that the (timely) development of market share cannot be forecasted because it does behave chaotically. In contrast to it, the final market share does not show chaos, because it is a conserved quantity. In section 3 of chapter III the diffusion model of marketing is scrutinized. (It is a standard model to estimate the market development of a new product) Already in 1993 a proof of chaos in the diffusion model has been published. Here it is shown that the seemingly chaos effects are due to a mathematical misunderstanding. Correcting these mistakes makes the chaos effects disappear and makes the use of the diffusion model much simpler. In the 4th section of chapter III it is proven that a typical financial market is never stable and will therefore show chaotic behavior. The corresponding statement 3 on page 66 is the reason why speculation can never lead to profit in the long run and is therefore identical to gambling.

1. Initial situation in operations

As already stated quantitative forecasts are essential in business and economics. Plans for the future are derived from it and quite often they are the essential inputs for a long term strategy. Results of management science and especially the use of computers made it much easier to make quantitative predictions. So far it is completely analogous to weather forecasts. Be it in physics, business or economics, it is assumed that the world is deterministic. Note that if something is not deterministic, it will happen by chance. If something is deterministic, people try to find out which causes have which effects. In physics these are the basic laws. They are essentially known. In business and economics mankind is far away from it. Partly there are some hints for it and partly there are at most rules of thumb. So if a model in business or economics does not lead to a proper result, there are two possible reasons for it:

- The input parameters are wrong or inaccurate.
- The prediction method is not good or less accurate in the long or short run.

In their respective effects both reasons are undistinguishable. This may be considered as a sad fact. But it stresses the point that it is very important to consider the range of validity of any result. Every physics' major learns in his or her first year that <u>every</u> measured quantity has a margin of error. Without at least estimating it, a measured quantity is completely worthless. Including this margin of error, any physical model gets its range of validity. It may be considered self-evident and indeed it is. The real surprise is that it is hardly used outside physics. Even in economics it is rarely found. Sometimes there are predictions like "GDP will grow between 2.6 % and 3.2 % next year." But even that is normally not a real margin of error. Most often it is the spread of, say, five institutions which have made a forecast independently. Especially in business there is sometimes a misinterpretation of statistics. There are sayings like "this strategy is 80 % right," which means, it is quite apt in 80 % of the cases, but in 20 % of the cases it may be complete nonsense. Such interpretation is twofold wrong. Firstly, there are not, say, 100 identical business situations and 80 are going in this way and 20 in the other. Secondly, if a model or procedure is sometimes wrong, it is generally wrong. One counterexample is enough to falsify a theory!

From this it becomes pretty clear that chaos effects are rarely seen in business and economics up to now. If something has a completely different outcome though the initial setup was almost identical, people might just say that this is a situation where the model is not correct.

2. The effect of mathematical chaos on business models

Chaos is when small causes have big effects. More than a century ago mathematicians found functions like f(x) of [3] on page 17. Normally, if the argument of a function (here x) varies very slightly, f(x) (the output) varies also very little. But with the above mentioned function we have a situation where $|f(x) - f(x + \varepsilon)|$ will always stay on average ½ completely independent of ε. This is especially surprising for very small ε. If x is assumed an input parameter and f(x) an output parameter, the output is strictly deterministic from a theoretical point of view. However, as stated in the previous section, every measured quantity has a margin of error. Even if it were arbitrarily small, the output would always vary by $\pm$ ½. So in all practical situations the output can only be determined within this accuracy (which is quite large because f(x) stays between 0 and 1 here. In other words we have a margin of error of $\pm$ 50 %). So if something is governed by such a function, accurate predictions are not possible. As I will show later the above mentioned f(x) describes the amount of money in a (strangely managed) bank account (for details see subsection 3.1 of chapter II). This is arguably the oldest prove for chaos in a business situation (here accounting). However, it has been suggested by a mathematician (Peitgen (1984)) rather than an economist or even management science professional.

Please note that such effects may appear also outside the world of formulas and mathematics. As an example consider an annual conference. The deadline for submitting papers may be June 30, 12 hours (sharp). At exactly 12 o'clock on June 30 we have a very chaotic behavior. The paper will appear one year later or earlier depending on whether it is submitted a millisecond later or earlier. In business situations planning is made by applying some mathematical formulas and some rules like for example the deadline is… So it is extremely likely that chaos will develop. In contrast to that, Peitgen (1984) was the first bringing up an example from accounting. Maybe because it is quite artificial, the business world never applied it. Later Weiber (1993) presented that a standard model of quantitative market research (here the diffusion model) may show chaos and is therefore of limited use. Such news should have been a "show stopper" for the diffusion model. Astoundingly it was not. Even today it is still used in the same way. It is even more astounding that the reported chaos effects of Weiber (1993) disappear completely, if one uses the correct mathematics in the diffusion model. This has been shown for the first time in section 3 of chapter III in this thesis. (Using the correct mathematics makes it even dramatically easier to apply the diffusion model. But even today the wrong mathematics is used, which is at most an approximation.)

Besides the approaches of chaos in business of the last paragraph, it was Grabinski (2004) who noted the consequences of mathematical chaos on business and economics. Later Grabinski (2007, 2008) gave some more detailed hints and proved the effect of chaos in the optimal warehouse if there are only two customers. This was the starting point of this thesis. In section 1 of chapter III, I have extended it to three and more customers.

Summary and Conclusions

If chaos is present, it would be useless to go on with calculating values for chaotically varying quantities. Appel (2011, 2012) used conserved quantities to describe chaotic situations in financial markets. This was my starting point to prove the conservation laws in business and economics (see subsection 3.2 of chapter II). In all the business situations where I have found chaos I also looked for conserved quantities and showed explicitly that they are not varying chaotically.

3. Conclusions and further research

As a conclusion, there are many more areas where chaos effects should be scrutinized. Here I will stress the implications for the applied research done by companies, banks, consultancies, etc.

Though there may be many areas, I will focus on two: The effects on software (development and usage) and the implication for companies' strategies.

3.1 Effects on Software

In my thesis I show effects of chaos on logistic operations like warehouse planning and planning as done in marketing. I perform all calculations by deriving the algorithm first. Then I perform them individually. In business situations all this is done by readymade software. Enterprise resource planning (ERP) systems such as SAP are used for the basic and daily business. But also for optimal warehouse location there is readymade software available. Therefore software developers and users should be aware of potential chaos effects.

In the main part of my thesis I mention the long computation time (CPU time) for calculations in the presence of chaos. For creating figure 9, for example around 100 hours of CPU time on a powerful workstation have been used. This is mostly due to the high accuracy which is necessary within a chaotic regime. Therefore standard software (be it Excel or SAP) will never get true results in the presence of chaos. This is no great obstacle at first glance only. Within chaotic regimes any result is completely useless. Therefore it is not necessary to know it exactly. It is apparently random. Whether these "random numbers" are produced by highly accurate calculations or partly by the effect of inaccurate software might be of academic interest only. However, standard software produces final results such as an amortization time. And only this final number is of interest for the user. In order to calculate this number very many steps are performed by the software being used. (Typically the steps are very manifold; else the use of software wouldn't be necessary) Maybe some of the numbers being produced in between show chaotic behavior. Then the result (be it chaotic or not) is most likely wrong.

In order to see this point more clearly I consider a strangely managed (chaotic) bank account. For more details see subsection 3.1 of chapter II. There figure 12 on page 30 shows the account balance after each period. It appears to be a complete random number. To calculate this random number is numerically challenging. If I had performed the calculation with for example Excel the picture would look similarly random. But the details would be completely different. (A similar effect is displayed in figure 7 on page 21, though it is a purely mathematical one.) After 120 periods there are for example 817,959.10 € in the account in reality while an Excel calculation yields 580,250.92 €. Of course, nobody is interested in the details of figure 12. However, as argued in subsection 3.1 (of chapter II) the total money taken out of the account (plus interest) is a conserved quantity. Therefore, it must not fluctuate chaotically. And indeed it does, what is displayed in figure 13 on page 31. It is essentially the sum of the chaotic points of figure 12. Of course, I have calculated it with very high accuracy. Had I not taken into account such high accuracy, figure 13 might also show chaotic behavior.

So there are some important learnings for software developers:

- For each step within the algorithm it should be clear whether chaos effects are possible or not.
- The software should be tested for chaotic effects in the final result and all internal results.
- The best way is to use conserved quantities only wherever possible.

For software users it might be even more important to be aware of chaos. As mentioned above using Excel, to calculate the account balance in subsection 3.1 (of chapter II) is very simple but completely wrong. In all kind of planning or forecasting one should always assume a margin of error within the input data. One may assume a Gaussian distribution with some typical width. (This is completely analogous to the procedure in subsection 2.2 of chapter III.) Please note that the distribution of data is normally not a Gaussian one and may depend on the particular situation. However, this otherwise so important remark is not of relevance here. One may always choose a Gaussian distribution (or any other convenient one). Varying the input data in that way will lead to a variation in output data. If the output varies with the same distribution, and its width takes the same order of magnitude, there is no chaos. Nevertheless, such analysis is necessary, even if chaos is exclude because for example conserved quantities are used. This analysis yields the margin of error in the output data. And without knowing the margin of error any result is of no use whatsoever.

If such analysis give rise to chaotic effects (when the output distribution is random), then any result should not be considered at all. There is a chance that the software algorithm is chaotic only. One may try with software with a completely altered algorithm. However, most likely such software will not be available. And one has to consider that chaos dominates in reality. Then planning is not possible here, and the future is as undetermined as the long term weather. Using such data nevertheless is foolish. Here nothing is worse than wrong data. Please note that the use of chaotic data happens quite often. Even worse some enterprises pay to get chaotic data. In Britain weather dependent industries such as ice cream producers pay quite some money to metrology departments to get a long term weather forecast. The departments are happy to take the money and deliver something which is as good as throwing dice.

Please note that detecting chaos rigorously may require a lot of computing power. It is normally not available to the software user. Furthermore, there is no chance to increase the accuracy arbitrarily within most software programs such as Excel. However, sometimes it is possible to reduce the accuracy. If the software for example takes sixteen digits accuracy one may reduce it to for example eight digits. All results (being chaotic or not) should stay identical within the first few digits. If not, something is wrong there. If they are identical, it is no proof of sufficient accuracy. But is a hint that the accuracy is most likely okay.

To summarize, there are the following recommendations for software users:

- Try to use and consider conserved quantities only.
- Always vary the input data and see what the output data are doing. Besides chaos, the margin of error is derived in doing so.
- Perform all calculations twice: With standard accuracy and half the standard accuracy. Expect at least very similar results.

To close this subsection I have some remarks for software users and developers alike. Varying input data within a Gaussian distribution is straight forward, but can be extremely tedious. Doing it "by hand" is excluded in most business situations. Therefore one should have some extra software producing a Gaussian distribution from a string of information in the particular format. Up to my knowledge such software is not available on the market. Therefore one should produce it oneself or order it from a special software maker.

For further developments of standard software an automatic chaos check is very desirable. For some software (for example Microsoft Project) there are add-ons available, but by no means for all. Producing such add-ons should be a lucrative business.

3.2 Implications on business strategy

This thesis is predominantly on operations rather than strategy. However, the most important implications may be on strategy. Essentially I show that many forms of long term planning are impossible due to chaos which is analogous to the impossibility of long term weather forecast. But strategy is a plan or a way how to reach some long (or medium) term goal. And this goal is a result of long term planning. Therefore people speak of "strategic planning" if they mean long term planning (normally for a couple of years in contrast to say a monthly plan).

So, most strategies might be affected by chaos, except for the trivial ones. Trivial strategies I call situations where the outcome is clear beforehand. Consider for example the Swiss defense industry. They have one customer only: the ministry of defense. As set in the constitution, the ministry of defense has to buy Swiss. Normally they are producing advanced weapon systems in license. So it comes as no surprise that the Swiss defense industry is able to plan its revenue with an accuracy of ± 1 % over five years. So there are no chaos effects, but the entire strategic planning is maybe considered a farce anyway.

To see the chaotic effects in "ordinary" strategies I will present an example. In order to find a suitable strategy, companies are planning for example revenue of A, B, and C over the next couple of years. "A" may be one's own revenue and "B" and "C" the corresponding revenue of two important competitors. (Alternatively A, B, and C may stand for the revenue of three different products) There are six possibilities:

1. A > B > C   oneself is biggest, and B is bigger than C
2. A > C > B   oneself is biggest, and C is bigger than B
3. B > A > C   B is biggest, and oneself is in the middle
4. C > A > B   C is biggest, and oneself is in the middle
5. B > C > A   B is biggest, and oneself is smallest
6. C > B > A   C is biggest, and oneself is smallest

Depending on which of the six scenarios is most likely for the future, up to six different strategies are necessary. In the case of five or six, it is probably better to be a quality leader, while in the case of one or two, price leadership isn't a bad advice. So one may try to plan or better estimate the revenue of A, B, and C for the next ten years in order to choose a proper strategy. This is a typical business of a manager trying to find an apt strategy. (However, real models of a strategic fit are normally more advanced and involve more quantities to be planned. But this simplification is sufficient to see the point.)

It was Grabinski (2007) who noted that *every* quantity has a margin of error and so does a planned quantity like the revenue. Using a model of a simple differential equation Grabinski (2007) found

Summary and Conclusions

that the uncertainty (= margin of error) in a plan increases exponentially over time. In other words, the uncertainty of a two year plan is not double as high as for a one year plan, but it grows exponentially. Mathematically, the uncertainty "u" (measured in for example ± percentage of error) increases like:

$$u = u_0 \cdot \left(e^{t/\tau} - 1\right) \qquad [\,1\,]$$

Here "t" is the time. (t = 2 years in case of a two year planning period.) τ is a constant (dimension time) which describes a typical period of change. In the internet business changes are fast. Therefore τ is small there, typically smaller than a year. In the case of the Swiss defense industry, market changes are very slow. Therefore τ is very big there (many years). The pre-factor u_0 determines the strength of the uncertainty. While τ is determined by the environment (market), u_0 depends on the skill of the planner or the quality of the planning procedure. Please note that such growth in uncertainty is also an effect of chaos. In an *identical* world all outcomes (for example development of revenue) are identical. However, there are many input parameters, some are not known and even the known ones are not known precisely. They vary by a certain (small) amount. And this variation causes a substantial effect over time. Therefore, it comes as no surprise that the uncertainty grows exponentially in time in [1], cf. [7] or figure 8.

Of course, all this is also true for the plan of the future revenue. The revenue is not given by a (curved) line. Rather the line grows exponentially in width as it is shown in figure 1. Up to t = 5 years one can be sure that A > B > C always holds. Then effects of chaos start to take over. For t =

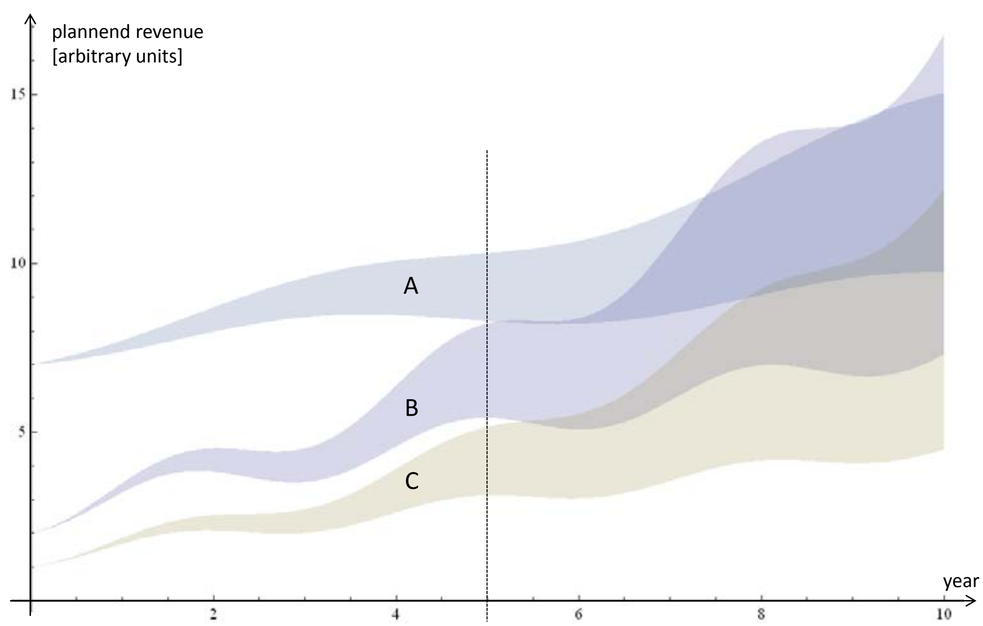

Figure 1: Plan of revenue for A, B, and C including margin of error

Summary and Conclusions

10 years it is completely unclear which revenue will be bigger or smaller. Within this example it would be possible to make a strategy for five years, but it would be impossible to choose a strategy for the next ten years. Depending on the situation this is a more or less severe problem. Especially if the strategy demands some long term decision, one should be very cautious. It is not rare that different strategies involve some different investments in bricks and mortar which run on a time scale of say 20 years. In the case of figure 1 it would be impossible to choose a strategy under such circumstances. One could rather throw dices.

If long term strategies are derived from quantitative parameters like within the so called strategic fit, one should always do the following:

- Find the margin of uncertainty for all relevant input parameters.
- Determine the timely horizon where uncertainty does not matter (five years in figure 1).
- Compare this time with the time a strategy must be valid (for example amortization time) and decide whether a strategy does make sense of not.

A merger or acquisition is maybe one piece of a strategy. Because an acquisition is generally spoken an investment, all the above is true and important for it. However, mergers and acquisitions show some additional problems due to chaos and especially the results of this thesis. It is easily seen by considering figure 28 which shows the market value and the (real) conserved value backed through the future cash flow and its market price. Firstly, a typical listed company is vastly overpriced compared to the cash flow it will produce eventually. Secondly, its market price varies chaotically. So it is impossible to choose the right moment to buy it. (In the same way it is impossible to find the right time to place your money on “23” on the roulette table.)

From this perspective mergers and acquisitions are like gambling. It might be one part in the explanation why they fail more often (about 2/3 of all cases) than succeed (1/3 of the cases). Please note that both effects vanish if one company pays for the other by granting a certain amount of stocks of the new (merged) company. However, though there is quite often a stock exchange within acquisitions, at least partly it is paid by cash. Furthermore, a company having lots of cash is badly advised not to use it. And there are also big privately owned companies which want to acquire another (publicly owned) company. Due to my arguments above they are almost certainly excluded from a successful takeover. As an example one may take the German Schaeffler KG solemnly owned by Maria-Elisabeth Schaeffler and her son Georg F. W. Schaeffler in 2010. They tried and succeeded to take over Continental AG a big German tire maker. Of course, they paid cash for Continental’s stock. Foreseeable, it ended in a financial disaster because they had to pay far too much per stock. Essentially they paid 10 billion € for something which was valid 2 billion € one year later.

I. Chaos and conserved quantities

The goal of this chapter is to explain the mathematical background of chaos and conservation laws, as far it is necessary to understand the main part of this thesis. For experts in mathematics and science the following two sections may be common sense, though the thoughts about conservation laws are less common than often claimed.

The third section about business situations, chaos and conservations laws is essentially new. Though chaos and business situations were very much in favor around the 1990s, the validity of this works are pretty limited (see for example Allen (1990), Nonaka (1988), Weiber (1993)). Arguably Grabinski (2004) was the first to deliver scientific results in this area. The considerations about conservation laws in business or better management science are fundamental and new. They are the direct necessary extension of the work of Gutenberg (1998). He predicted that there *exists* a function determining the outcome in business and economics. He never said how this function looks like and more importantly what are its variables. This is a very fundamental statement. But because of its generality it has no direct application up to now.

1. Definition of chaos

The word *chaos* has its linguistic roots in the word *Χαοζ* of ancient Greek. Its original meaning was "empty space." With the Roman influence the meaning changed to "shapeless", in the sense that "God created heaven and earth from chaos." Order has been taken into place. Therefore we have for chaos the modern meaning of "disorder" or "irregularity." However, though this definition is in accordance with the mathematical chaos discussed here, it is not what most people understand by the word chaos. They have something in mind like a "chaotic day." Which means a terrible day, a day full of stress. Though this is a common meaning of chaos, it is absolutely not what I understand by chaos in this entire thesis.

In mathematics functions play an essential role. Consider for example the function $f(x) = 3x$. For $x = 10$ one finds $f(10) = 30$. Changing x by one percent from 10 to 10.1 will yield $f(10.1) = 30.3$, and 30.3 is exactly one percent bigger than 30. There are other examples which give less trivial result. For $f(x) = 3x^2$ one finds $f(10) = 300$ and $f(10.1) = 306.03$. While x turns one percent bigger, f(x) turns 2.01 % bigger. For $f(x) = 3x^{10}$ a one percent increase in x means a 10.4622 % increase in f(x). Though functions change more or less with their arguments, a sufficiently small change of the argument will give a desired small change of the functional value. Mathematically spoken:

$$\forall\, x, z \quad \exists\, \varepsilon > 0 \quad \perp \quad |f(x+\varepsilon) - f(x)| < z \qquad [\,2\,]$$

Though most functions fulfill the condition from [2], some do not. At first glance that looks like a mathematical toy with no application whatsoever. And surely it was, when such functions had been discovered over 100 years ago. (One of the pioneer mathematician in this area was Poincaré in around 1890.) However, the condition of [2] is far from being some mathematical toy. From science to business we have formulas or functions to predict the outcome. If somebody saves € 100 for twelve months, he or she will have € 1233.07 if the interest rate is 5 % annually (compound interest is included in the calculation). If the proceedings should be € 10 more (€ 1243.07), how much should the interest rate be? This is very easy to predict (6.478 % annually) though the detailed formula for it is complicated. Everybody dealing with finances should be able to answer such questions. However, if the "interest function" would not obey [2] such questions would be impossible to answer. (An example for a bank account with exactly such properties will be given in section 3 of this chapter.)

In order to see an example consider the seemingly simple function

$$f(x) = \lim_{n\to\infty} \frac{1}{2} * (1 - cos[2^n \cdot arccos(1 - 2 \cdot x)]) \qquad [\,3\,]$$

The function is defined for 0 < x < 1. Within this regime, 0 < f(x) < 1 always holds. The limit does formally not exist; however it is understood in a way that there is always a sufficiently large "n" so

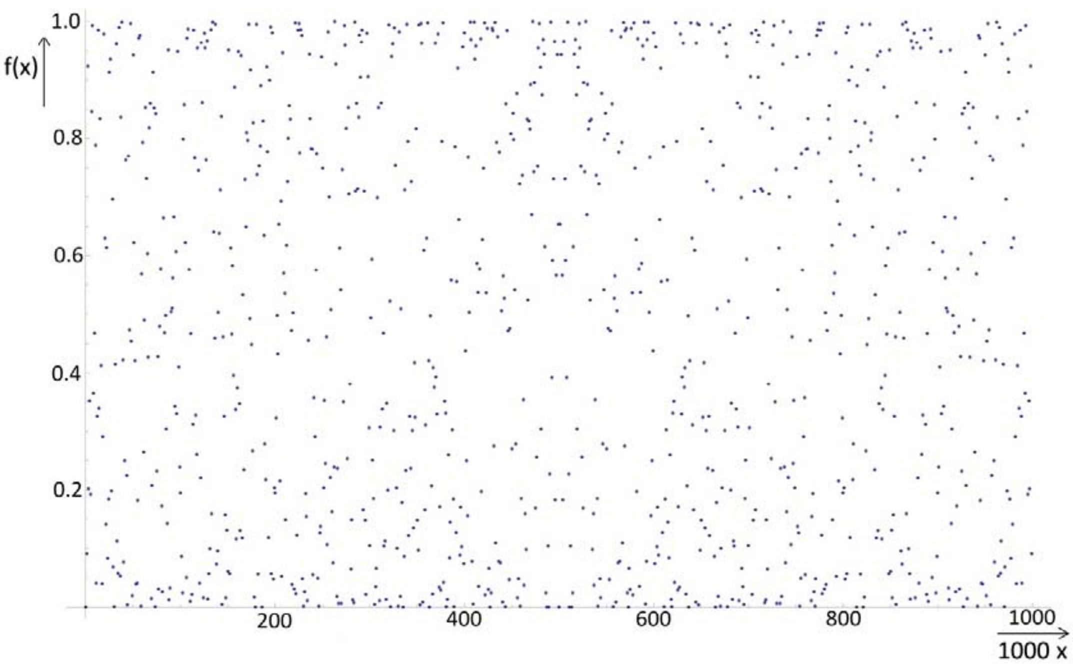

Figure 2: Chaotic function f(x) (n = 10000) (discrete plot with 1000 points, else the line would fill everything)

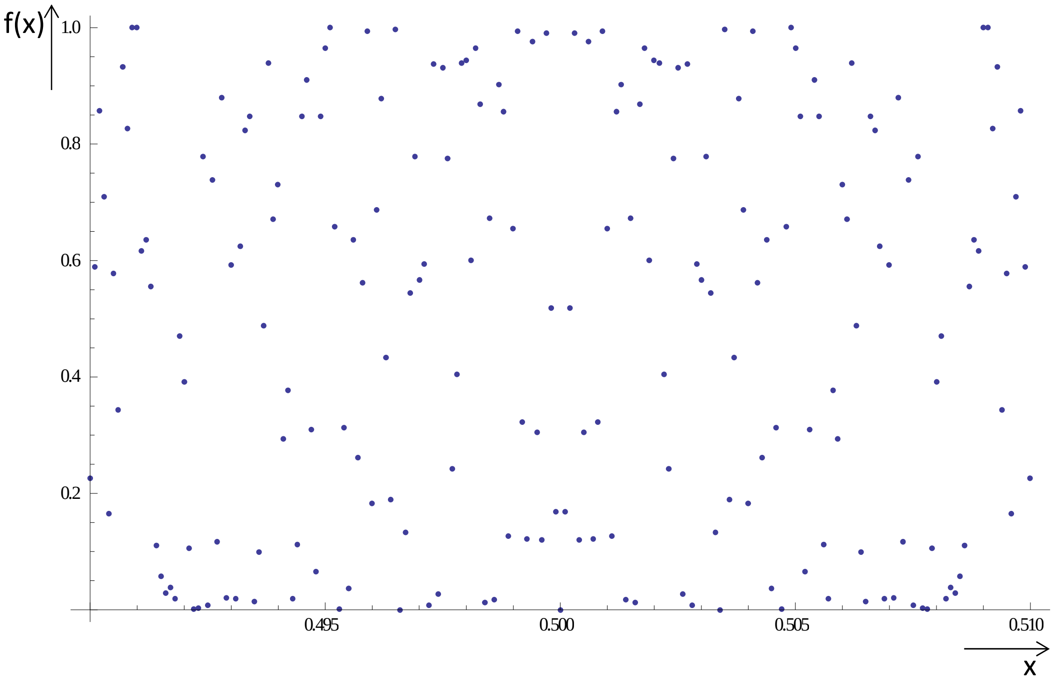

Figure 3: Chaotic function f(x) (n = 10000) (discrete plot with 1000 points, else the line would fill everything

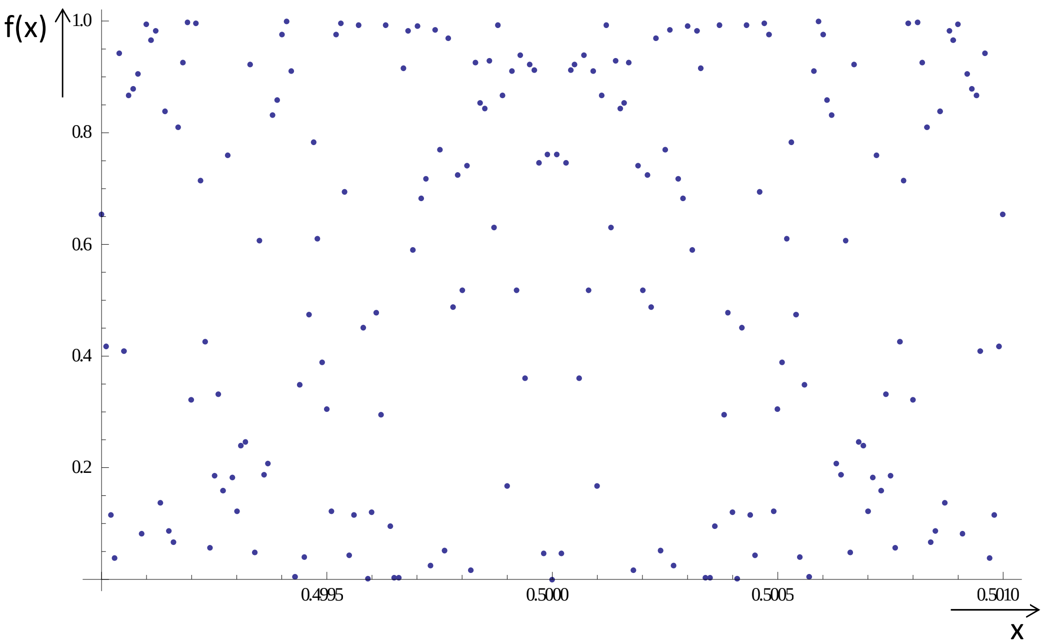

Figure 4: f(x) around x = 0.5 $\pm$ 0.01 (n = 10000) (discrete plot with 200 points, else the line would fill everything)

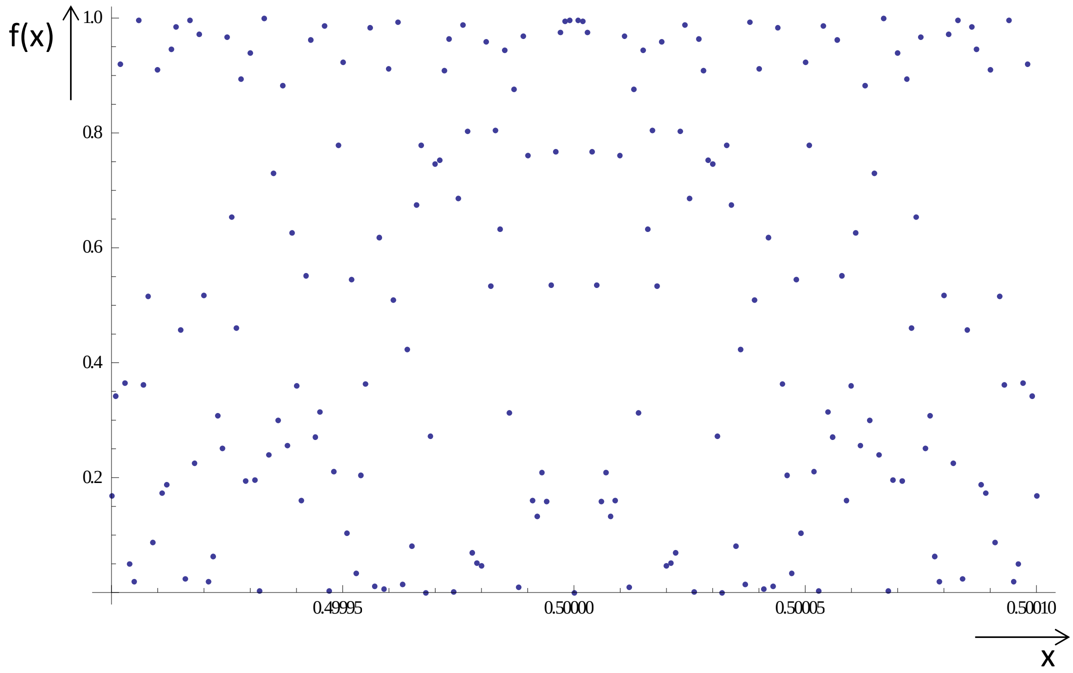

Figure 5: f(x) around x = 0.5 ± 0.0001 (n = 10000) (discrete plot with 200 points, else the line would fill everything)

that [3] does not hold. To see the strange behavior of [3] set for example. n =10000 and go from x = 0 to x = 0.999 (step 0.001). The 1000 f(x) values are displayed in figure 2.

Figure 2 looks like a complete random pattern. Please note that the numerics is far from being trivial. As a rule of thumb one has to calculate with as many digits as “n” (here 10,000). It is even more striking that one can go finer and finer, and still there is the same pattern. This has been displayed in figure 3, figure 4, figure 3: Chaotic function f(x) (n = 10000) (discrete plot with 1000 points, else the line would fill everything, and figure 5. f(0.5) = 0 for $n \in N > 1$. In the very near vicinity it has a complete different value. If something is governed by such a function, no prediction is possible. Actually, the initial value x = 0.5 is supposed to vary by around $\pm 10^{-3013}$ (three thousand and thirteen zeros after the decimal point!) in order to make a decent prediction in this case. Needless to say it is beyond any accuracy in this world. Such situations are therefore considered unpredictable. Of course no computer or software will help. The only way out in this situation is to find a completely different description. A good way out is to use conserved quantities (see next section). But one has to accept that some things cannot be predicted. Ignoring that is not only a waste of time. If results are created in chaotic situation and taken at face value, management effort is spent, but the predictions are as good as throwing dice.

Besides [2] up to now the essential definition of chaos is that small causes have tremendous effects. Such definition is by no means bad, especially in business situations where more often than not no formulas are given. Nevertheless it does make sense to look for more rigorous definitions. In what follows I will therefore discuss some standard mathematical definitions.

The maybe most often used definition of chaos is the occurrence of “white noise.” If the input values show some (narrow) distribution (little noise), the output will show an infinitely wide distribution (all frequencies, white) if chaos is present. This is graphically displayed in figure 6.

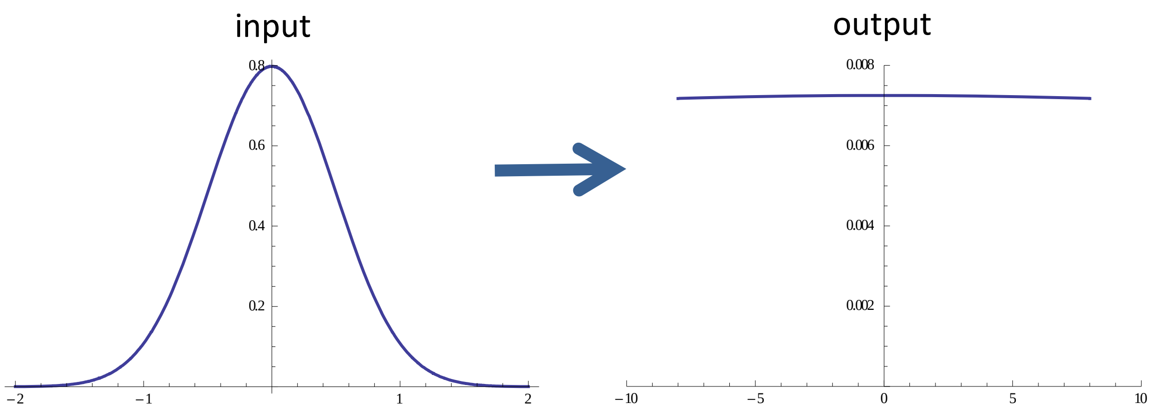

Figure 6: Distribution of input and output values in the presence of chaos

For a fully developed chaos an arbitrarily narrow Gaussian distribution will turn into a complete random distribution. In, for example, figure 6 one easily sees that this is the case. Of course, one may talk of a continuous transformation from non-chaotic behavior into fully developed chaos. The distribution becomes wider and wider. Such considerations are very useful because in the real world there may be almost never fully developed chaos. Even in figure 4 there is no fully developed chaos. Only for $n \to \infty$ it is truly chaotic. With n = 10000 as plotted there the situation becomes quite regular on a scale of 10^{-3013}. So strictly mathematical there is no chaos; for any practical purpose however it is extremely chaotic. Such an analysis from input to output values like in figure 6 should be performed in any kind of analysis or planning. If the output distribution is very wide, the entire consideration is completely worthless, be it because of chaos or whatever reason. Unfortunately comparing distributions of input and output is rarely done, especially in management (cf. Grabinski (2007)). Even an ordinary error analysis is almost unknown.

Another litmus test for chaos is the so called Lyapunov exponent. In order to understand it better, I will give another example of a chaotic function: *the logistic map.* Arguably it is the most scrutinized object in chaos theory by mathematicians and scientists. The so called logistic map is defined by the following recursion formula:

$$f_n(a,s) = a \cdot f_{n-1}(a,s) \cdot \left(1 - f_{n-1}(a,s)\right) \quad and \quad s = f_0 \qquad [\,4\,]$$

For the starting value s, $0 \le s \le 1$ must hold. "a" is a “factor of strength” with $1 \le a \le 4$. Starting with s = 0 leads to f(n) = 0 for all “n”. It is a so called fix point. Starting with s = 1 also leads to f(n) = 0 for all n > 0. Depending on “a”, there are other “strange” points like s = ½ for a = 4. Please note that the standard literature such as Schuster (1984) defines the logistic map as a function of “n” with a parameter “a” and a starting value “s”. It is of course the “same function” with the same properties. I have chosen the form given in [4] because it is closest to an ordinary function describing some timely development over “n” periods. Indeed [4] describes the capital development after “n” periods on a strangely managed bank account. I will come back to it in section 3 of this chapter.

But the most interesting behavior of the function of [4] is given if “a” is sufficiently large. Up to a = 3, $f_n(a,s)$ is a pretty ordinary function. After that point it will jump between two values depending on “n” (bifurcation). At “a” about 3.45 the two values split again. If “a” becomes 3.5699456… or bigger the situation becomes chaotic. For sufficiently big “n” the slightest change in “s” (or “a”) will change $f_{n>>1}(a,s)$ beyond recognition. Please note that there are many plots of

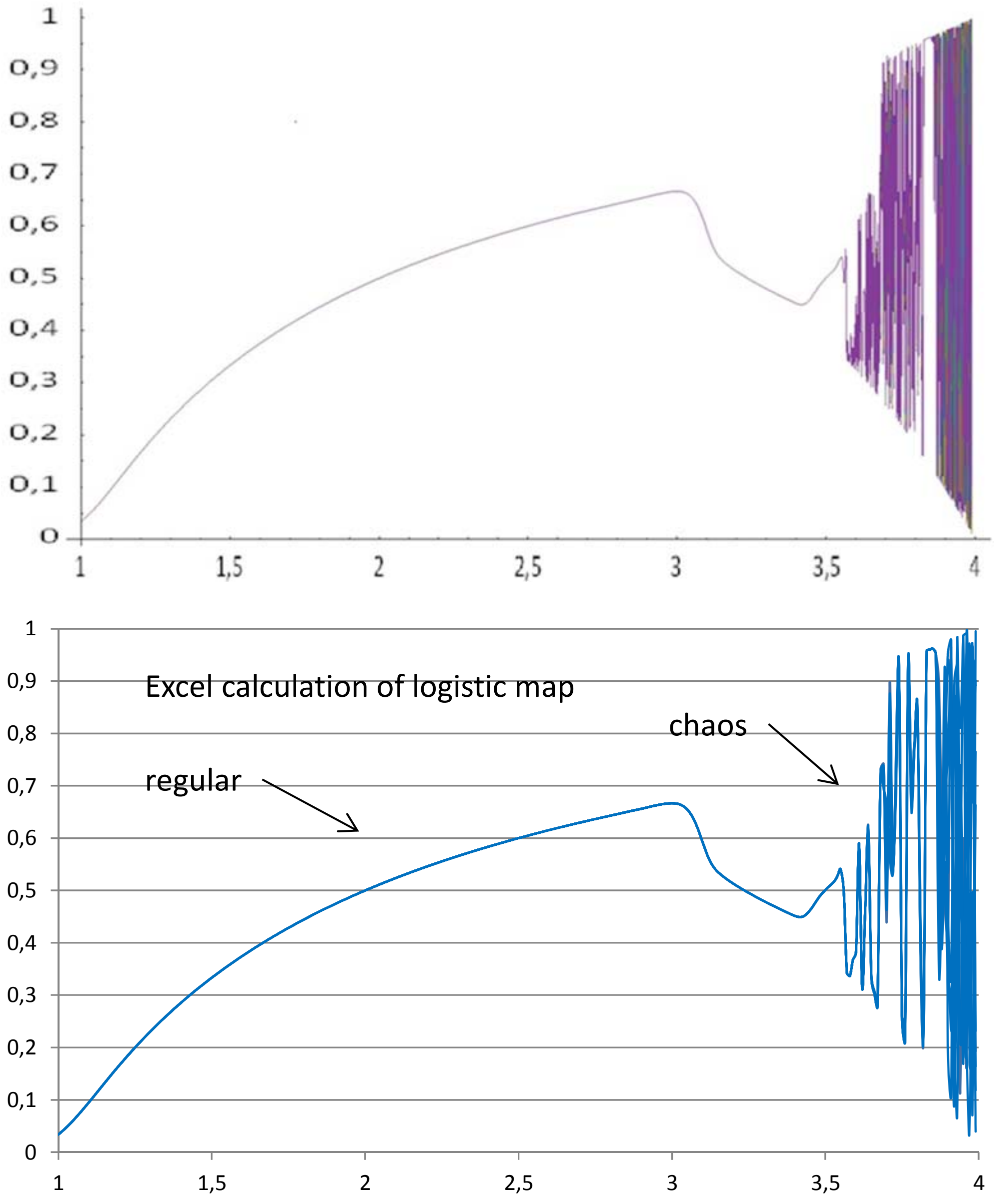

Figure 7: Logistic map, n = 25 iterations, ten plots for ten slightly different start values

this behavior in the internet or textbooks like Schuster (1984) or Grabinski (2007). Though the general form is okay the details are properly wrong. This is because the function is so chaotic that a numerical calculation becomes difficult. Calculating f_n with Excel is at first glance pretty simple. But for n > 20 and s > 3.6 the results are random numbers rather than the true values. One roughly needs to carry 2^n digits of accuracy for "n" iterations in [4]. For n = 25 this means an accuracy of over 30 million digits. In figure 7 I have plotted f_{25}(a,0.333328), f_{25}(a,0.333329), f_{25}(a,0.333330), f_{25}(a,0.333331), f_{25}(a,0.333332), f_{25}(a,0.333333), f_{25}(a,0.333334), f_{25}(a,0.333336), and f_{25}(a,0.333337) over "a" from 1 to 4. So the starting values just changed by a thousandth of percent. The exact calculation used about 20 hours of CPU time and 13 GB Processor memory. I have included the Excel plot in figure 7 for comparison (It took less than one second of CPU time). The rough picture is the same but the details are different. It stresses again that even todays computers have a hard time to deal with chaos. Be it as it may, figure 7 shows that for a > 3.6 no prediction is possible though only 25 time intervals were considered and the starting value had been known within 10^{-3} percent or one Cent in 1000 Euros. Please note that even for a > 3.6 there are some areas where the exact and Excel calculation are identical. This is due to the fact that even after a =3.6 there are some small islands where the function remains regular. Within those islands it is easy for Excel to calculate the true values.

It is maybe useful to note that the logistic map of [4]and the example of a chaotic function in [3] are close allies. More specific, the function of [3]is identical to the logistic map for a = 4. The proof of it is quite simple if one uses complete induction and the fact that $\cos(2\alpha) = 2\cos^2\alpha - 1$.

But now I want to come back to the Lyapunov exponent. Because mathematical considerations of chaos are best understood by iterative functions like the logistic map, the definition of the Lyapunov exponent starts with an iterative function of the form:

$$f_{n+1} = g(f_n) \qquad [5]$$

In the case of the logistic map g(x) = a x (1-x). In chaotic functions adjacent points like $f_n(x_0)$ and $f_n(x + \varepsilon)$ increase in distance dramatically after n iterations. This is displayed graphically in figure 8:

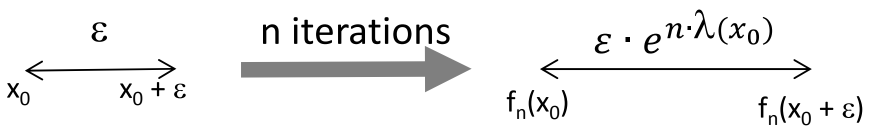

Figure 8: Definition of λ

It is always possible to write the growing distance in the form of figure 8. It is a definition of a function $\lambda(x_0)$. Please note that λ also depends on n, g, and ε up to now. From figure 8, λ can be written in the following form:

$$\lambda(x_0) = \frac{1}{n} \cdot Log\left|\frac{f_n(x_0+\varepsilon) - f_n(x_0)}{\varepsilon}\right| \qquad [6]$$

Taking now the limit $\varepsilon \rightarrow 0$ will transform the quotient within the absolute value into a differential of it. Taking than the limit $n \rightarrow \infty$ leads to the final definition of the Lyapunov exponent:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} Log \left|\frac{df_n(x_0)}{dx_0}\right| \qquad [\ 7\ ]$$

Please note that the λ [7] is different from the one of [6]. Just for simplicity I have used the same symbol. f_n is the n-th iteration. In general it is a very complex function of a chain of g-functions: $g(g(g(\ldots x_0)\ldots)))$. Besides the definition of λ the interpretation of it is very interesting.

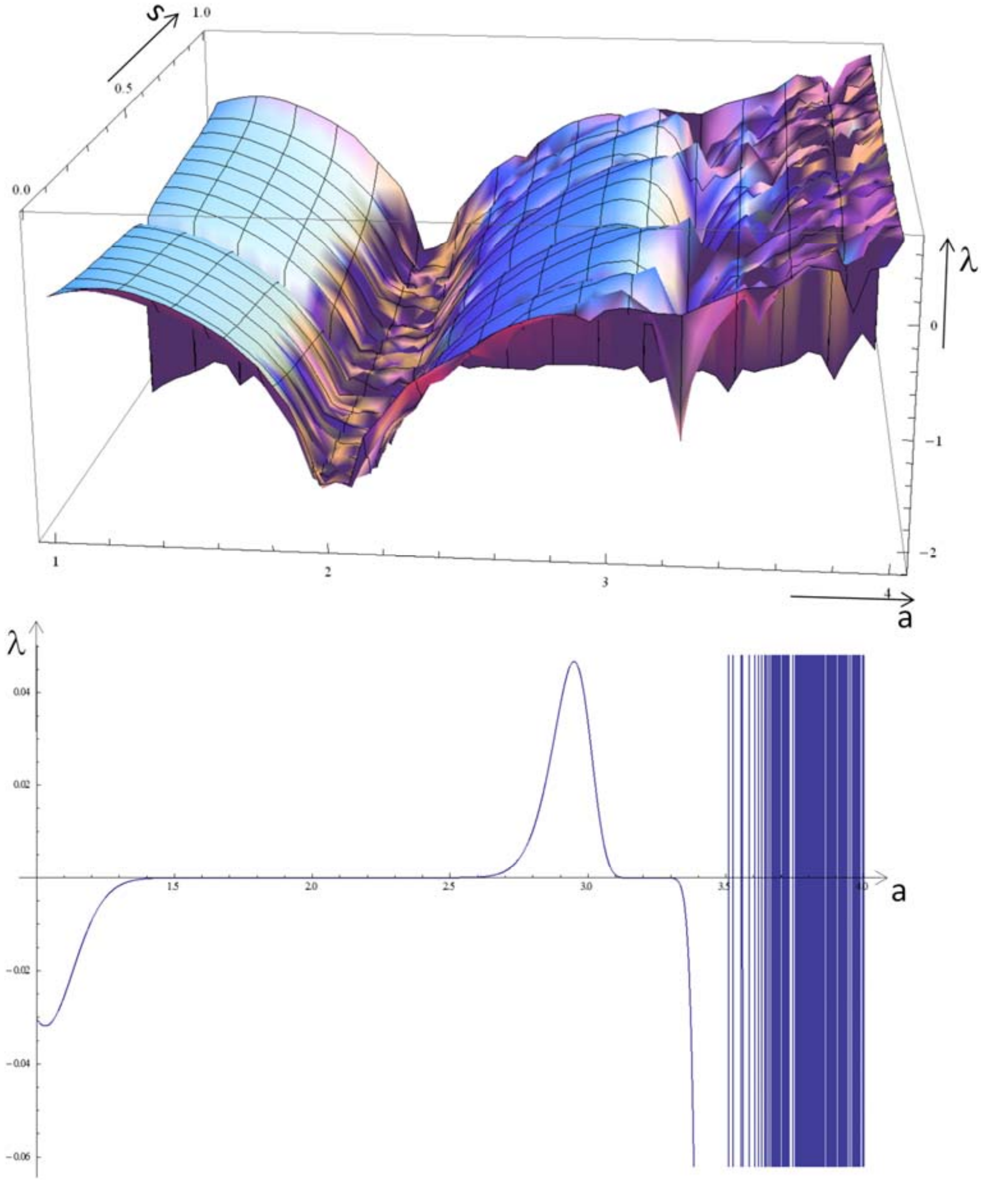

Figure 9: Lyapunov exponent λ(s, a) and λ(0.8, a) for logistic map (n = 20)

It is the exponent for exponential growth of the distance of two adjacent points. If $\lambda < 0$, the distance between the two points becomes exponentially smaller by each iteration. This is a complete regular behavior. However, if $\lambda > 0$ the distance is growing exponentially. This is called chaos. Please note that even without taking the limit $n \to \infty$ a Lyapunov exponent does make sense. It still shows whether the distance between two adjacent points is increasing (chaos) or decreasing (no chaos). Furthermore, λ always depends on the starting value x_0. That does make sense. Consider for example the logistic map of [4] where $x_0 = s$. For $s = 0$ chaos will never show up. The same is true for $a = 4$ and $s = ½$. In figure 9 the Lyapunov exponent of the logistic map as function of starting value s (or x_0) and the prefactor "a" has been plot. In addition, there is a plot of the Lyapunov exponent for starting value $s = 0.8$ over "a" for clarification. λ depends on "a" and "s." However the typical transition to chaos is given at around $a \approx 3.6$ almost independent of the start value s. Furthermore, there are areas of regular behavior ($\lambda < 0$) even for $a > 3.6$ and vice versa. Please note however, that I have used $n = 20$ in the logistic map only. Hardly a perfect approximation for $n \to \infty$. But already for $n = 25$ the calculation of figure 9 needs more than 300 GB in processor memory, which was not available. But even with that approximation λ oscillates rapidly for $a > 3.6$. This stresses the point that the limit [7] (for calculating the Lyapunov exponent) does not necessarily exists.

2. Conservation laws

In science and especially physics conservation laws are used. The best known is perhaps the conservation of energy. More intriguing may be the conservation of mass (as measured in kg). Here, conservation laws of science are not discussed for their own sake and by no means in detail. The purpose is to learn something for business and economic situations (see next section). Therefore two questions should be answered:

- What makes conservation laws so interesting?
- What have conserved quantities to do with chaos?

The short answers to each question are that conservation laws imply equations of motion. With it we can predict the future. And conserved quantities can never show chaotic effects. Both are definitely interesting for business and economics. It is the main business for managers and economists to predict the future and act accordingly today. Furthermore, the last section has shown that chaos may spoil everything. Therefore it is important to understand the questions to be able to understand their answers. In physics, equations of motion are well known, cf. Newton (1711). In business and economics nobody knows the “equations of motion.” In this thesis conservation laws for business and economics are developed for the first time. For a better understanding, conservation in physics is considered first.

Consider a sphere as indicated in figure 10. The total mass in the sphere may be M. It consists of many small pieces of mass m_i. The mass inside is $M = m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7$. Without

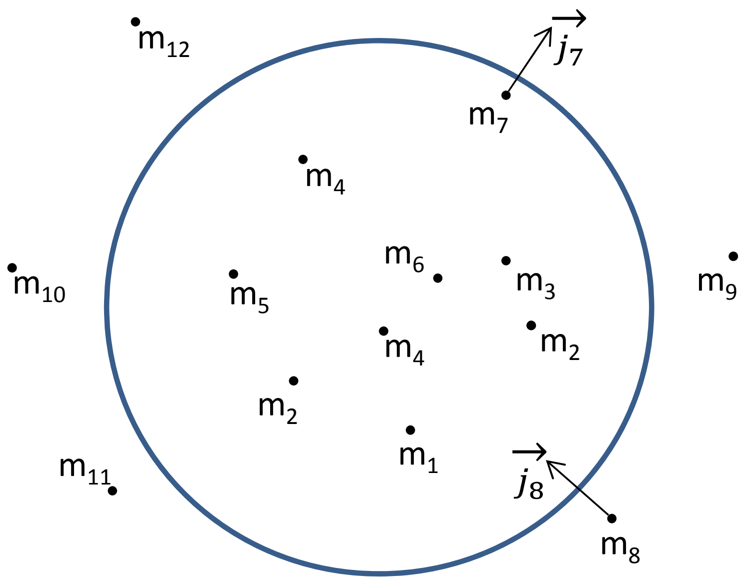

Figure 10: Graphics for mass conservation

any masses going in or out, the total mass M stays constant. It is conserved. It cannot be created or annihilated. As also indicated in figure 10, m_7 is about to leave the sphere indicated by a mass current $\vec{J_7}$. Because the mass current flows in a certain direction it is a vector, indicated by the little arrow over the symbol. The same is true for the mass m_8. It is about to enter the sphere as being indicated by the mass current $\vec{J_8}$. So the change of the total mass M over time in the sphere is opposite to the sum of the mass currents. If the dot over M denotes the timely change (or its derivative with respect to time) one may write the formula:

$$\dot{M} + \sum_{i=1}^{n} \vec{J_i} \cdot \vec{S} = 0 \qquad [\ 8\ ]$$

Here $\vec{S}$ is an unit vector (length 1) perpendicular to the surface. In that way straightly outgoing masses are counted positive and incoming negative as it should be. Instead of considering separate mass points one may also take the continuum limit. Then the mass is distributed continuously. Furthermore one may consider an arbitrarily small piece of volume dV. In this piece the mass will be dM =ρ dV, where ρ denotes the density and density times volume is mass. Getting from dM back to the entire mass M one has to take the volume integral. The summation sign in [8] turns into a surface integral over the closed surface S. Doing so one gets:

$$\iiint_V dV\,\dot{\rho} + \oiint_S d\vec{S} \cdot \vec{j} = 0 \qquad [\ 9\ ]$$

Applying the Gaussian integration theorem, which says that a surface integral over a closed surface can be written as volume integral over the divergence $\iiint \mathrm{dV}\,\mathrm{div}\,\vec{\mathrm{a}} = \oiint \overrightarrow{\mathrm{dS}} \cdot \vec{\mathrm{a}}$ with $\mathrm{div}\,\vec{\mathrm{a}} = \vec{\nabla} \cdot \vec{\mathrm{a}} = \partial_x \mathrm{a}_x + \partial_y \mathrm{a}_y + \partial_z \mathrm{a}_z$, transforms [9] to[9].

$$\dot{\rho} + div\,\vec{j} = 0 \qquad [\ 10\ ]$$

[10] is a differential equation for ρ = ρ(x,y,z). It is the equation of motion. In classical mechanics there are three other conservation laws: One for momentum (mass times velocity), one for angular momentum (rotation frequency times moment of inertia), and one for the energy as mentioned above. Because this thesis is not about physics, I will not go into detail. But some historical remarks are interesting here. The conservation of momentum leads to a similar but more complicated differential equation like [10]. It is the so called Navier Stokes equation, which is the key equation in fluid dynamics. Claude Louis Marie Henri Navier and George Gabriel Stokes proposed this equation independently in 1827 and 1845, respectively. Though the conservation of momentum had been well-known for over hundred years at this time, Saint-Venant who proved the equation first, used a very different way to prove it. He used Newton's equation to do it. Up to my knowledge, it was the late Nobel laureate Landau who first used conservation laws to derive

the Navier Stokes equation in the 1950s, almost hundred years later. Deriving equations of motion for arbitrary systems rather than just fluids is younger still. It was Liu in the 1970s (for example Liu (1978)) who did it. He called his approach (euphemistically) the standard procedure. Though it applies to any physical system from fluids over solids up to neutron stars, it is far from being standard even today. Though this hydrodynamic approach as it is also called is extremely powerful, it is rarely used. Even in today's physics lectures the Navier Stokes equation is derived by using Newton's equation more often than not. Being useful for so many physical systems, it is astounding that it should not be a powerful tool in business and economics. As this thesis shows, it is useful and powerful in business and economics. On the other hand, seeing the history of conservation laws in physics, it may take hundred to two hundred years until they will be common sense in business and economics.

In the last section I have shown that it is useless to observe quantities which fluctuate chaotically, and it is impossible to predict the future of these quantities. The good news for, for example, a manager from this is that he or she has not to care about these quantities. The urgent question is what else can be done? What other quantities can be used? Such questions led to the work of Appel (2011). There he found already that the market value (or better market price) may fluctuate chaotically and is of no use whatsoever. Therefore it was necessary to define a conserved value, because it cannot fluctuate chaotically. The proof is quite simple. If a quantity is conserved, any change of it must be caused by a change in initial value (or something else). (In the case of the mass it is a mass current) Therefore an arbitrarily small change cannot cause a macroscopic effect. In other words chaos is prohibited or [2] is always fulfilled. Of course, this is true for anything in the world, not just physics. The only prerequisite for this "anything" is that it is governed by cause and effect or causality. And causality is generally assumed. However, even if causality were violated in some cases, it is still a valid law for everything from science to management. This is because things not obeying causality are completely useless to be considered for analysis or predictions. So I can conclude this section with the following statement:

Statement 1: Use of conserved quantities

- → **If a quantity has the ability to behave chaotic, it is a waste of time to consider it. Instead conserved quantities are the only reasonable variables to describe, analyze, or predict in this situation.**

Please note that this statement is not *new*. It is known in physics for a long time. For other situations like business and economics it had been suggested by Grabinski (2004) for the first time. Here it is explained and rigorously proven.

3. Business situations

Now the theory of the last two sections, which originated mainly in physics, should be transferred to business and economics. Here it is done in a general form. In the next chapter (Examples in business and economics) it will be done in detail for some situation with detailed results.

In the subsection "3.2 Conservation laws in business and economics" I will transfer the general content about conservation laws of the last section to business and economics. This is however far from being just an application. Here, a new conservation law in business and economics is derived comparable to the conservation of energy in physics (actually it exists for the same reason as in physics).

3.1 Chaos in business and economics

As stated already in section 1 of this chapter, the (modern) mathematical considerations of chaos started over hundred years ago. The first striking effect in the real world manifested in the 1960s when Edward Lorenz discovered chaos effects in the mathematics of weather forecasts. He proved that even the little eddies of a butterfly wing have an effect on the start of, for example, next month's rain. Obviously that makes long term weather forecast impossible. This effect is commonly referred to as "butterfly wing effect." It is interesting to note that, for example, the average temperature of the earth, be it the spatial or timely average, are very predictable and non-chaotic. They belong to a conserved quantity (here: energy). The same is true for the average precipitation (again spatial or timely average). The reason is here the conservation of mass. At least from that point in history it should be clear that chaos will most likely show effects in business and economics, where predictions and forecasts are the *bread and butter* business. However, such interpretations were not made for quite a while in contrast to science and engineering. The 1970s and 1980s were full of discoveries of chaos effects in these areas. There were useful applications like machines with a chaotic force distribution in order to have bearings to be worn out evenly. The end of the 1980s and early 1990s marked the hype for chaos in business and economics. However, these works did not come to the point and are far away from the rigorous descriptions of this thesis (see for example Allen (1990), Nonaka (1988)). Other discussed seemingly important effects, but did make horrible mistakes (Weiber (1993)). The latter work was the origin of the third section in the next chapter ("3. Diffusion model in marketing"). The only exception was the work of Peitgen (1984) who reported about chaos effects in a bank account. I will discuss it in detail further below. (Please note that Hans-Otto Peitgen was a pure mathematician and neither an Economist nor a business professional.)

Chaos and conserved quantities

The first who noted seriously about effects of chaos in business and economic situations was Grabinski (2004). There an example of traffic flow was given. Later possible effects on logistics were discussed (Grabinski (2007, 2008)). They formed the origin of this thesis. One can only guess why it took so long that chaos effects were included in business and economic considerations. Some thoughts about it can be found in Grabinski (2007). Firstly, for a thorough understanding of chaos (beyond "little causes have tremendous effects") one has to be fluent in modern pure mathematics (from Hausdorff dimension to information entropy). Secondly, colorful pictures of fractals in chaos brought it into an esoteric rather than scientific corner. Thirdly and most importantly, chaos is a "disruptive" effect in business and economics. It tells that something is not possible for principle reasons. But it does not necessarily tell how to do it correctly. (The conserved quantities help to overcome the problem.) And especially managers are far away from admitting "I don't know it." or even worse "There is no answer to it." It tells what can be done and what not. It even shows that some managerial work of today's managers is a complete waste of time and money. Even more striking, quite a huge chunk of the financial industry delivers no value for mankind (cf. Appel (2011, 2012)). All this should be seen as an encouragement and stress the importance of this thesis.

Now I will come back to the example of banking as mentioned above. Let's supposed somebody has a savings account and periodically takes away some of the interest it gained. The main question to answer is how much money will be in that account after a certain time in dependence of the money invested originally. This seems to be a standard (and trivial) question for a bank manager. Because the customer is a free person, he or she may withdraw whatever he or she wants. So one may set two rules for the bank account:

- The currency is calculated in a way that the bank account contains always something between zero and one. (For example: If € 123,456.78 are in the account one will call it 0.12345678 million Euros.)
- After each interest period the original capital times the capital including interest is withdrawn. (For example: 0.8 was in the account and it gained 12.5 % in interest, then $0.8 \cdot 1.125 - (0.8 \cdot (0.8 \cdot 1.125)) = 1.125 \cdot 0.8 \cdot (1 - 0.8) = 0.18$ will be in the account.)

It is a clear and unambiguous rule. The first one is just stated in order to keep the second rule simple. It is easy to show that the account will never become negative. Furthermore it will always stay between zero and one. Though the rules might sound strange and unmotivated, some crazy millionaire might just want it that way. And every bank manager will probably say that it is very easy to predict the financial status of this account.

If somebody puts € 1.00 in such account and the interest is 300 % within one period there will be € 817,959.10 in the account after120 periods. (Please note that 300 % of interest sounds high, but alternatively one may take a long period). Starting with one cent less (€ 0.99) there will be € 865,051.03 in the account after 120 periods. This is 5.76 % more, though one started with 1 % less. Starting with 1 % more (€ 1.01) leads to € 493,965.31 or 39.6 % less. Showing a graphic stresses the point even more clearly:

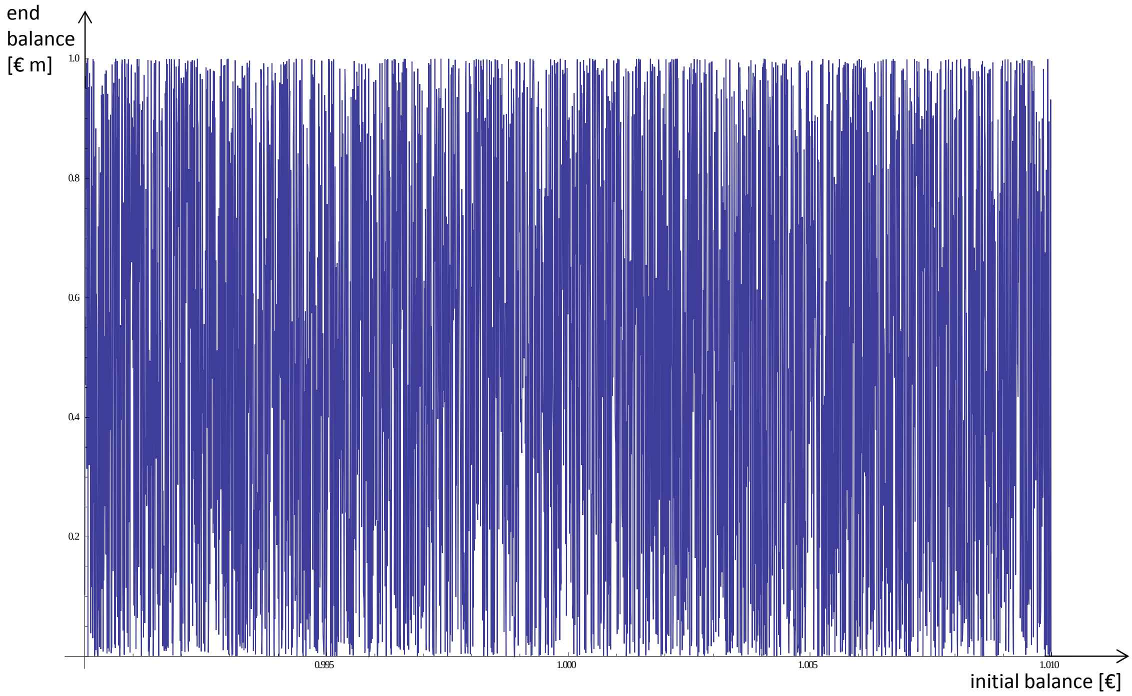

Figure 11: Account balance after 120 periods, starting capital between € 0.99 and € 1.01

As one sees from figure 11, for any very slight change in the initial capital the ending balance is jumping (almost) randomly between zero and one million Euro. It appears to be unpredictable. Especially the reverse question is difficult to answer: Find an initial capital which leads to for example € 123,456.78 after 120 periods! To find this solution is almost impossible even with today's powerful computers, at least if one uses the iteration and not the formula from [3]. It is also intriguing to look, how much money is in the account after each period, if one starts with one

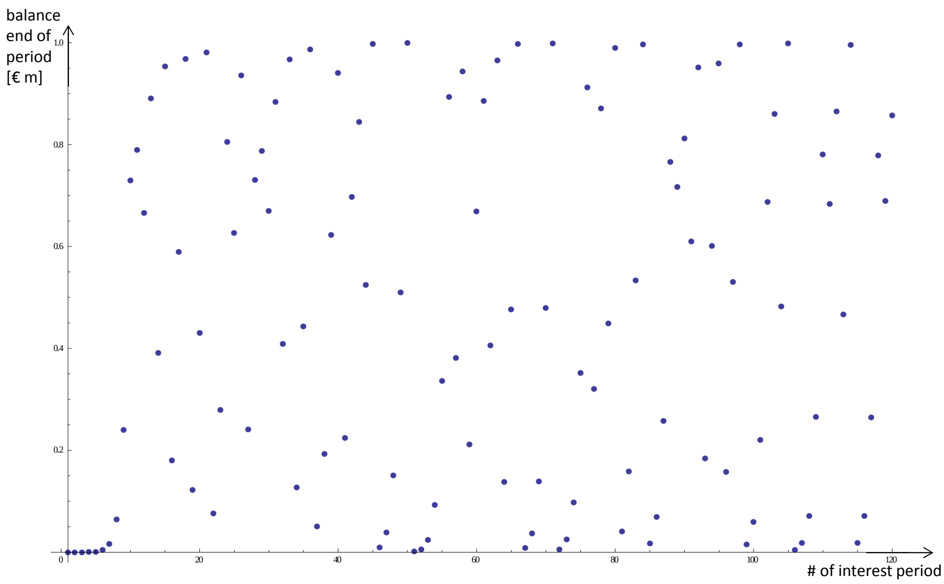

Figure 12: Account balance after each period

Euro. This is shown in figure 12. Only during the first few periods the behavior looks regular. The rest of the 120 points are scattered around almost randomly. As stated above, this is probably the first example of chaos in a business situation.

The mathematical proof of chaos is pretty simple here. As most readers might have recognized already, the formula for the account balance after each period is nothing else than the logistic map of [4]. The interest rate is "a – 1" in [4]. From this it is clear, that I have chosen such a high interest rate in order to find chaotic behavior. The particular interest rate of 300 % (or a = 4) have been chosen in order to use [3] instead of [4]. (This reduced CPU time dramatically. However, Excel is still not able to perform the calculations.)

The main point of this example is that there are business situations, where a standard parameter (here account balance) is totally useless due to chaos. The immediate question is, what parameter should be used to describe this process? The answer is given by statement 1 of the last section: It must be a conserved quantity, because such quantities cannot show chaos effect. A conserved quantity is something, which cannot be changed by a snip of a finger. In this example the capital gained due to interest is a conserved quantity. That it is the only one, will be shown in the next subsection. That it is a conserved quantity, has to do with the fact that the capital in itself is a conserved quantity. Else it would be possible to create capital from nothing which is obviously impossible. (In other words, in a world where capital could be created from nothing, such capital would be worthless.) To prove this theory one may consider the capital extracted at each period. The sum of it including interest (here 300 %) is given by:

$$\sum_{i=0}^{119} (4 \cdot s_i - s_{i+1}) \cdot 4^{119-i} \qquad [\ 11\ ]$$

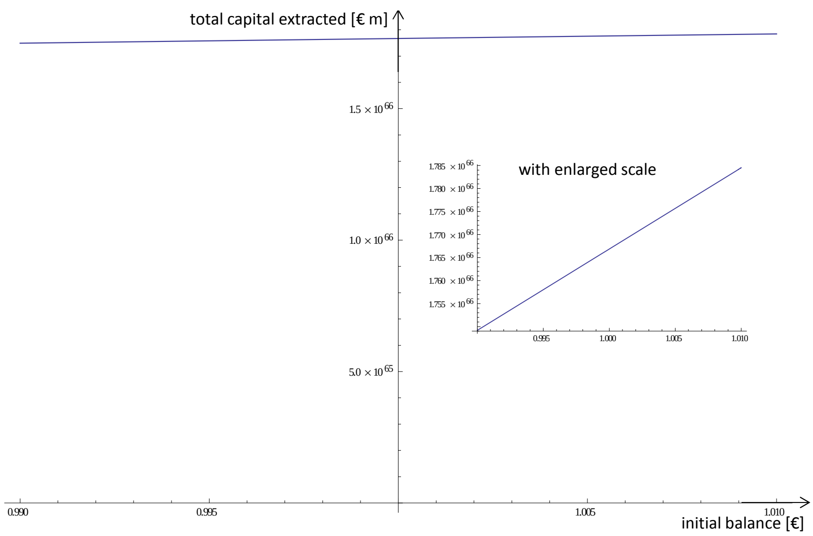

Figure 13: Total capital extracted in dependence of original balance

Here s_i denotes the account balance which varies chaotically, cf. figure 12. This sum must not behave chaotically, and indeed it does not as figure 13 shows. The enlarged scale shows a perfect linear behavior as it should show. So this example shows nicely how statement 1 holds.

3.2 Conservation laws in business and economics

In the second section of the last chapter I have shown, what conservation laws are, and that they are not affected by chaos. (A detailed consideration can be found in Klinkova (2017).) In the previous subsection 3.1 I have shown an example from accounting where chaos exists and a conserved quantity (capital gained) does not show chaos because it is conserved. The only thing left to do is to show the conservation law(s) of business and economics. This is easier as it sounds at first glance.

The starting point is Gutenberg (1998). His systemic approach of the 1930s is much elder than the reprint of 1998. There are various descriptions of it. The one useful here goes as follows. It exists a function L

$$L = L(q_1, .., q_i, ... q_N;\ \dot{q}_1, ...\ \dot{q}_i, ... \dot{q}_N; t) \qquad [\ 12\ ]$$

which determines the development over time of an operation (company, state, etc.). Here q_i is a variable and $\dot{q}_i$ its derivative with respect to time t. What are these variables is completely open. They may range from “capital invested” over “hours worked” to “technology used”. There may be a very huge number of variables (N >> 1). Furthermore, the function L is far from being known. Only its existence is demanded. In other words, tell me the status quo inside and outside an operation and the rate of change of the status quo, then the status of any later time is completely determined. The existence of a function L of [12] is nothing but demanding causality (in a quantitative sense). Therefore, systemic approach means nothing else than causality in a quantitative sense. There is hardly anybody who might deny it nowadays. Please note that if there were something without the causality of [12], any management of it would be completely useless.

The function L of [12] is of course far from being unique. Therefore, it is allowed to demand something of the function L. In physics the existence of a function is often demanded. (Because Gutenberg studied physics and chemistry first before he turned to business, he probably derived his systemic approach from there.) One example is the existence of a function S (entropy) in thermodynamics. Another one is the existence of a function L (Lagrange function) in classical mechanics. In both cases the demand of existence is nothing but demanding causality, which sounds almost trivial. However, in physics the existence of the function and its particular variables are demanded. In other words, if such and such values of these variables are known, the future is determined. This is a much more stringent demand, and it leads to many results. Here we do not know the variables, but nevertheless one can derive one conservation law. Be it as it may, as a learning from physics, it is smart to demand some extreme behavior of the function. Of course, one may demand whatever one wants. But some demands will lead to conclusions and others not. In the case of entropy S one demands that S takes a maximal value. In the case of the Lagrange function L some functional W should become maximal:

$$W = \int_{t_1}^{t_2} dt\, L(q_1(t), .., q_i(t), \dots q_N(t);\ \dot{q}_1(t), \dots\ \dot{q}_i(t), \dots \dot{q}_N(t); t)\ \ \rightarrow\ \ maximal \qquad [\ 13\ ]$$

Exactly this should be demanded here (as stated above without any limitation). t_1 and t_2 are the initial and final points in time over which all the variables will develop. And the development over time of the q_i is exactly the question being asked. If one would know this development, one could make exact predictions and all problems of management science would be solved. It will remain a dream for the time being. This is because no one knows the variables q_i and in particular not the function L. [13] defines a standard problem of functional analysis. For solving it, the same approach as going from [80] to [81] of the appendix can be used. Doing so leads to the following N (coupled) differential equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad [\ 14\ ]$$

They are the "equations of motion" for all business and economic situations. Not knowing L it is useless to try to solve it. But other important conclusions can be drawn from [14]. One may take the total derivative with respect to time of L of [12], use [14], and the product rule of differentiation:

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_{i=1}^{N} \dot{q}_i \cdot \frac{\partial L}{\partial q_i} + \ddot{q}_i \cdot \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial t} + \sum_{i=1}^{N} \dot{q}_i \cdot \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \ddot{q}_i \cdot \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial t} + \frac{d}{dt}\sum_{i=1}^{N} \dot{q}_i \cdot \frac{\partial L}{\partial \dot{q}_i} \qquad [\ 15\ ]$$

Up to now this is just pure mathematical rearrangement. However, the very general form of L in [12] is "too" general. This is because L must not depend explicitly on time t (only implicitly via the q_i). If it would depend on t explicitly there would be something strange. If all values of the variables are the same, but only the time t is different, every outcome should be the same. In other words, all laws of business and economics must not change over time. (The same is true for the laws of physics.) Therefore, $\partial L/\partial t = 0$ must hold. Then [15] can be written as a total derivative:

$$\frac{d}{dt}\left(L - \sum_{i=1}^{N} \dot{q}_i \cdot \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \qquad or\ \ L - \sum_{i=1}^{N} \dot{q}_i \cdot \frac{\partial L}{\partial \dot{q}_i} \quad is\ constant\ over\ time \qquad [\ 16\ ]$$

Chaos and conserved quantities

So we have derived a constant quantity or a conservation law. Comparing it to physics, this is the equivalent to the conservation of energy which stays constant because our universe is homogeneous in time. Please note that in physics there are at least two other symmetries or invariances. It leads to additional conservation laws.

Here nothing is known about symmetries or invariances within the variables q_i. So [16] is the only conservation law. Unfortunately nobody knows what $L - \sum \dot{q}_i \cdot \partial L / \dot{q}_i$ means because nothing is known about L. However, with some (small) assumption, one will find something. Up to now the only assumption was causality.

In order to find a less abstract interpretation one may take some further assumptions. (A detailed consideration can be found in Klinkova (2017).) My assumption dates back to Adam Smith who proclaimed that in business and economics everything is determined by a value v (in a monetary sense). So, if somebody is working, he or she might want to improve something in order to be more valuable in a monetary sense. In other words one will only consider activities which will increase or decrease some value v_i. With this assumption the variables q_i change into v_i.

Taking values v_i for the variables q_i in L one has $\mathrm{L} = \mathrm{L}(\mathrm{v_i(t)}, \dot{\mathrm{v}}_\mathrm{i}(\mathrm{t}))$. Without any limitation one may divide everything in smaller and smaller bits. In doing so the particular v_i becomes a smaller and smaller. Furthermore, one may change v_i arbitrarily slowly. Alternatively one may chose a time unit where $\dot{\mathrm{v}}_\mathrm{i} \ll 1$ always holds. In doing so L can be written as a Taylor series of lowest order:

$$L = a + a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + b_1 \cdot \dot{v}_1 + b_2 \cdot \dot{v}_2 + \cdots \qquad [\ 17\ ]$$

Using this in [15] the conserved quantity takes the form

$$L - \dot{v}_1 \cdot \frac{\partial L}{\partial \dot{v}_1} - \dot{v}_2 \cdot \frac{\partial L}{\partial \dot{v}_2} - \cdots =$$

$$a + a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + b_1 \cdot \dot{v}_1 + b_2 \cdot \dot{v}_2 + \cdots - \dot{v}_1 \cdot b_1 - \dot{v}_2 \cdot b_2 - \cdots = \qquad [\ 18\ ]$$

$$a + a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots$$

From [18] one sees that a linear combination of the values v_i is conserved. However, the only condition was that the v_i are small. Let $v_i' \equiv a_i v_i$. v_i' will be small, as long as v_i is sufficiently small. This transformation of value must also be applied to the timely derivative of v_i leading to newly defined b_i'. But these variables were cancelled out anyway. So we have found that the sum of the values (v_i') is conserved. This leads to the following important statement:

Statement 2: Conservation law of business and economics

→ **In all economic or business systems there is one strictly conserved quantity. Considering values and their changes implies that the total value is conserved.**

Chaos and conserved quantities

So it is explicitly proven that value is a conserved quantity. It was the (reasonable) assumption in the publications of Appel (2011, 2012). Together with statement 1 one sees that the only reasonable description of a complex and probably chaotic system must be done by using conserved quantities only.

The conservation of value has also another implication. The conserved quantity “value” as defined here is generally distinct from market price (sometimes also called market value). Obviously, the market price may change without changing anything else. Therefore, it cannot be a conserved and it must be different from value as defined here. Of course, it is not forbidden to consider the market price. It is even quite often necessary. But it is completely useless for planning or analyzing an enterprise if the situation is sufficiently complex. This becomes very clear if someone considers prices in the stock market instead of underlying company values (cf. section 4. Development of chaos in financial markets of the next chapter). Unfortunately most brokers do it just the wrong way.

III. Examples in business and economics

The last chapter explained the theoretical background of my thesis. In this chapter I will go into the details of business situation. It is an application of the theory of chapter two. I will show four examples. They are neither a complete set nor are they the most important examples. I have chosen these examples because it was possible to detect chaos which had not been noticed before.

Partly a lot of mathematics is involved in order to show the chaotic effects. As long as these mathematical considerations are not necessary to understand the main point, I have abandoned them to the appendix.

1. Warehouse location

To find the optimal warehouse position it is a standard problem in logistics. However, it is not the goal of this thesis to find new ways to solve that problem. (Though, especially in the appendix one may find some useful input for improving current algorithms) The goal of this section is to build on the works of Grabinski (2007, 2008). There it was shown that in the simple set up of just two customers the warehouse should always be placed at the biggest customer in order to minimize transport cost. However, if both customers become bigger and bigger, eventually two warehouses are cheaper than one. It would lead to zero transport cost and costs for two warehouses in contrast to just one. This is a macroscopic change, though the initial change in customer size may be arbitrarily small. And that means chaos, according to one of its definitions. If a customer size is around this particular value, planning warehouses becomes impossible, because nobody can forecast exactly how much a particular customer will consume in the future. On the other hand, one has to decide to build either one or two warehouses.

Though having two customers only may sound trivial, it is a situation where chaos occurs with all its consequences. The immediate question is: What happens if there are more customers? Does it become more chaotic (most likely) or is there some lucky averaging out? How is the principle way to show it? What are the consequences? What are the conserved quantities, and are they really non-chaotic (cf. statement 1)?

Please note that there is a lot of software available to find the optimal warehouse positions and to calculate associated transport costs. However, this software is completely useless here. Firstly, they perform numerical calculations only. One has to insert coordinates for the customers and one will get coordinates for the warehouse. However, I do need a formula in order to see where there is a chaotic regime. Secondly, all software does not deal with the question which warehouse should deliver to which customer. One has to state the number of warehouse first and also say which customer is delivered by which warehouse. Essentially it is solving the problem of one warehouse and a certain number of customers. (By looking at subsection 3.1 of the appendix it becomes clear why this is the case. Deciding which warehouse should deliver to which customer involves even numerically very complex mathematics.)

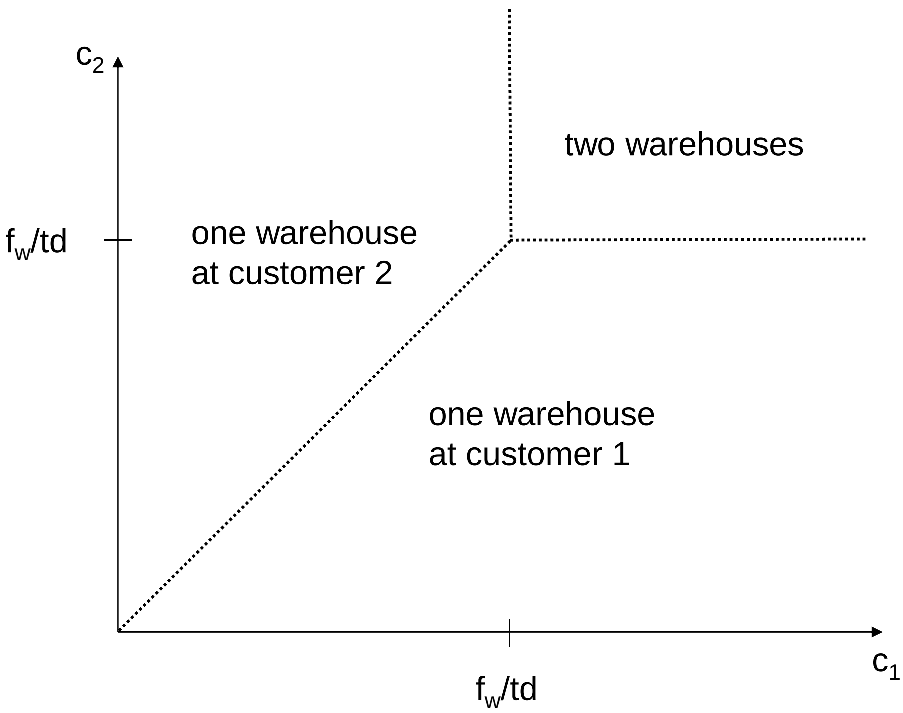

Figure 14: Chaos with two customers (source: Grabinski (2007))

In figure 14 the situation for two customers is summarized. Let them have consumption rates of c_1 and c_2 and being separated by a distance d. The specific transport cost is “t”, so that the cost transporting from customer one to two is $t \cdot d \cdot c_2$. In figure 14 the two consumption rates c_1 and c_2 are the labels of the axes. f_W denotes the fix cost of one warehouse. Depending on the consumption rates there is one warehouse at customer one or two, or two warehouses (one at each customer). The dotted lines separate regimes for one or two warehouses. Near these dotted lines an arbitrarily small change in consumption will shift the optimal warehouse position over a distance d or make the change from one warehouse to two. The dotted lines are (as all lines) one dimensional. The c_1-c_2-space is two dimensional. So, there is a one dimensional chaotic regime in the two dimensional consumption space.

For three customers, there should be a two dimensional chaotic regime within the three dimensional c space. I will show that it exists and how it looks like. So, one has to consider three customers C_1, C_2, and C_3 with consumption rates c_1, c_2, and c_3. For them, one has to calculate the optimal warehouse position and the corresponding cost (transport cost plus f_w). Then one has to consider two warehouses. Their optimal placement is pretty simple. The warehouses are supposed to be at the biggest customer and the second biggest as stated above. The total cost of this (= transport cost to smallest customer plus $2f_w$) must be compared to the total cost of one warehouse. Equalizing these two costs yields a function $c_3 = c_3(c_1, c_2)$ which defines the two dimensional surface in the three dimensional c space. Though this approach looks pretty straight forward, it isn't. Finding the optimal warehouse position for three (different) customers is possible numerically only. Therefore, in a first step one has to consider equal customers ($c_1 = c_2 = c_3 = c$). In a second step one assumes a deviation of it and expresses the problem in the variables c and $\Delta c = c_i - c_j$. Because one knows the solution for $\Delta c = 0$, one can build a Taylor series in Δc. This is what physicists call perturbation theory. Details can be found in the appendix starting with [56]. In doing so, one can solve the problem up to any order in Δc. Though this is really straight forward, it is incredibly messy. It is known as the Steiner Weber problem for three customers. Though Fermat found already its solution geometrically, I have never seen an analytic calculation of the optimal position (w_x, w_y). To the best of my knowledge, I have solved that problem in the appendix for the first time. The result is displayed in [53] on page 73. Even that very long formulas must be transformed by the transformation of [44]. Needless to say that this is possible, but whatever results, it will bring no conclusions can be drawn. There I have taken another approach:

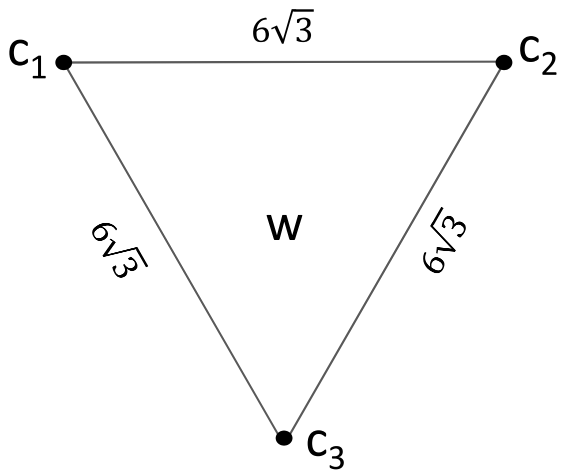

Figure 15: Three customers in a regular triangle

In figure 15 I have chosen three customers in a regular triangle. Though it is a limitation, it is hardly possible that this will cause more or less chaos effects. The customers may have an arbitrary but equal distance "l" from each other. I have chosen a length scale so that this distance is $l = 6\sqrt{3}$. This is by no means a limitation. (From the appendix 2 it might become clear why I have not chosen a length scale so that “l” is equal to one, which is of course also possible.) As long as the consumption rates of all customers are equal, the optimal warehouse position will be in the center as indicated in figure 15. For different consumption rates the position will vary towards the bigger customers. With bigger and bigger c_i the transport cost will become bigger and bigger. Eventually it is smarter to have two warehouses at the two biggest customers leading to total costs of $c_{two} = 2 \cdot f_w + 6\sqrt{3} \cdot t \cdot c_s$, where f_w is the cost of one warehouse, “t” is the specific transport cost (cost per distance and tonnage), and c_s is the consumption rate of the smallest customer. So, there is the situation where the costs for one warehouse and two warehouses are equal. At this set of c_1, c_2, and c_3 the warehouse position jumps from one warehouse somewhere in the middle to two warehouses at the biggest customers. A slightest change in the consumption rates c_i changes the warehouse position dramatically. In other words planning the optimal warehouse position within this regime is impossible. The formula for it is given by [65] of the appendix (Please note that [65] is valid for $c_1 < c_3 < c_2$ only. The remaining regime is easily constructed by symmetry arguments). As stated in the appendix, it is also possible that three warehouses are cheaper than two.

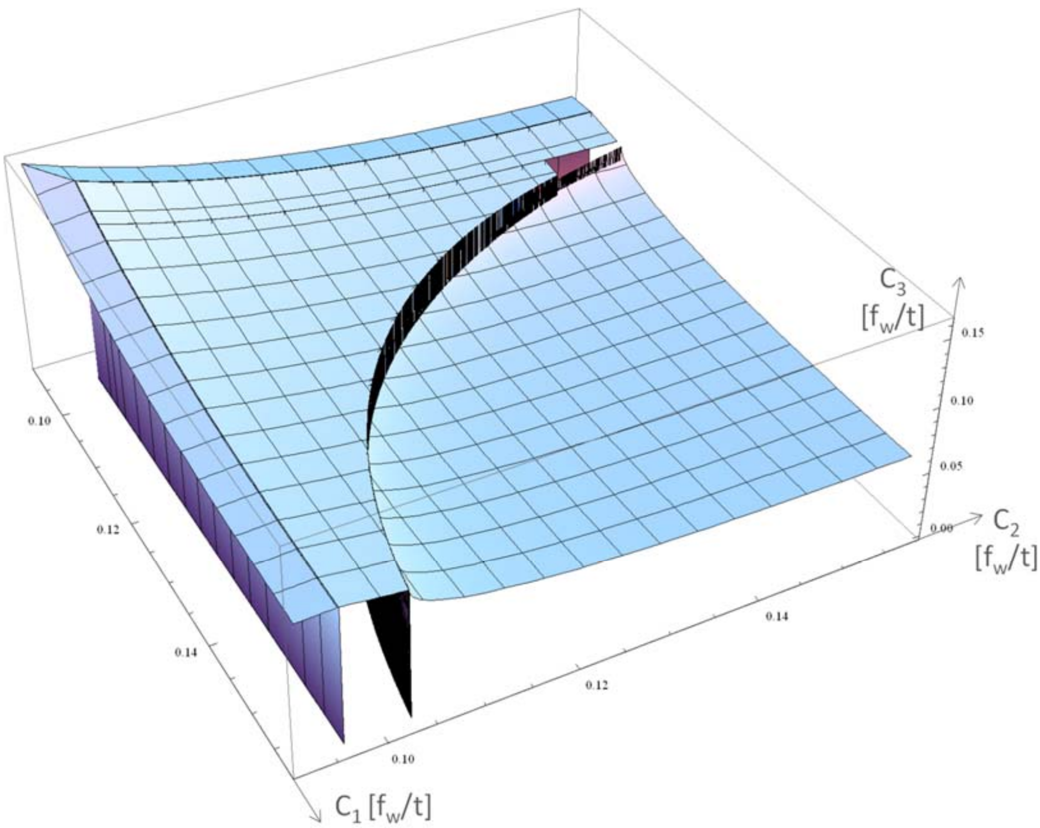

Figure 16: Surface of chaos: Surfaces of transition from one to two and one to three warehouses

In this case there is a (chaotic) transition from one to three warehouses. All this is summarized in the plot of figure 16. For sufficiently big values of c_1 and c_2 (and small c_3) as in the front right corner, there is a transition from one warehouse (below) to two warehouses at C_1 and C_2 (above). The same is true for the edges of small c_1 and c_2, respectively. In the middle where all c_i are similar there is a transition from one warehouse to three.

For sufficiently big values of the c_i with the smallest $c_i > f_w/(t \cdot 6\sqrt{3})$, three instead of two warehouses are always optimal. However, this is not shown in figure 16.

I have shown that the generalization from two customers to three leads to a more complex chaotic structure. Now one may go a step further and consider four customers. It should lead to three dimensional chaotic regimes in the four dimensional c_i space. Because nobody is able to think in four dimensions, I will show the principle only. The starting point is again a highly symmetric arrangement:

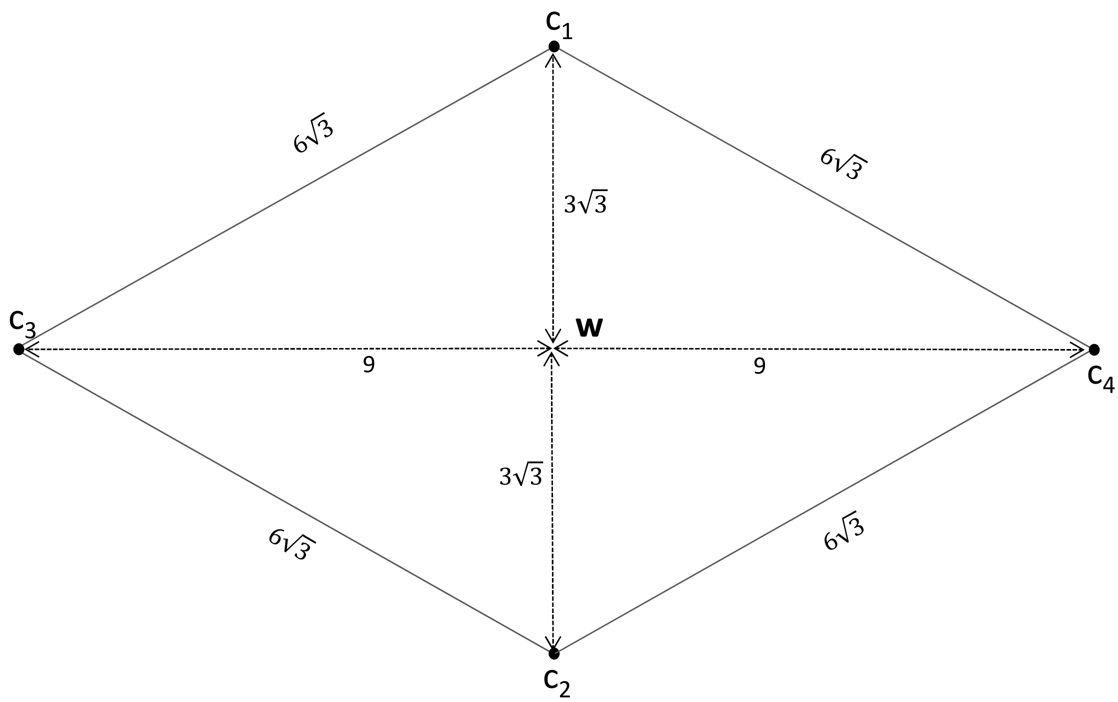

Figure 17: Four customers in symmetric arrangement

Now there are four customers in the diamond shaped arrangement of figure 17. Please note that the symmetry of figure 15 has been chosen in order to get simple formulas, though any triangle would have been possible. With four customers in an arbitrary quadrilateral, even for equal consumption rates a numerical solution is possible only. The length scales of figure 17 had been chosen so that the left hand side of figure 17 is identical to figure 15. Furthermore, I will assume equal consumption rates here or $c_i = c$. Such simplification makes the general logic easier, though it would be possible to start the same perturbation theory as with three customers. As long there is one warehouse only, its position must be right in the middle. So the total cost is:

$$c_{one} = f_w + (18 + 6\sqrt{3}) \cdot t \cdot c \qquad [19]$$

For arbitrarily small c this is obviously always the best choice. For arbitrarily big c, the optimum is to have four warehouses with $c_{four} = 4\ f_w$. It is the optimal solution if the (smallest) consumption rate is bigger than $\sqrt{3}/18 \cdot f_w/t$. The interesting question is what happens in between. For two warehouses two different positions are thinkable:

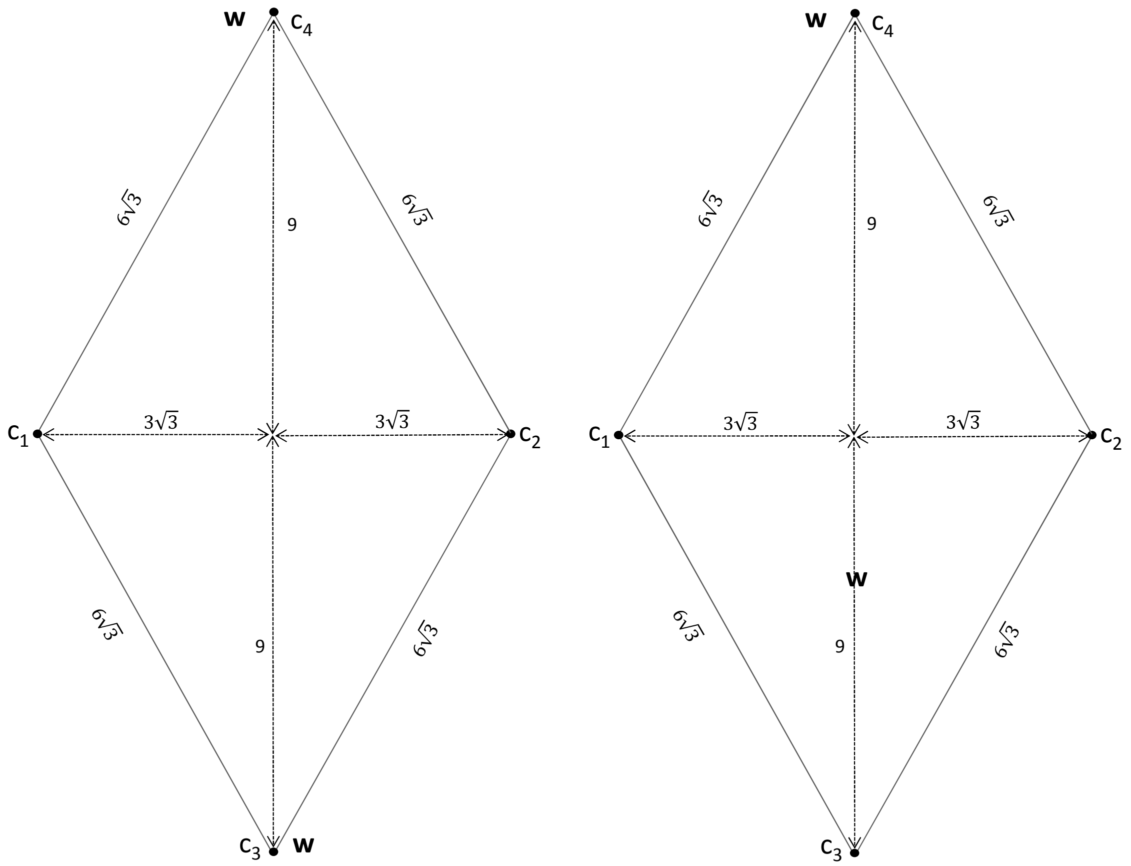

Figure 18: Possibilities for two warehouses and four customers

Either the two warehouses are at C_4 and C_3 or they are at C_4 and in the middle of the triangle of C_1, C_2, and C_3. All other arrangements are cost wise identical. This kind of arbitrariness has to do with the fact that all consumption rates are equal. If the consumption rates are almost identical but c_4 is the biggest and c_3 the second biggest, then figure 18 is the only choice. For the right hand side of figure 18 it is obvious which warehouse serves which customer. For the left hand side it is unimportant whether the two warehouses serve two customers each or one serves three customers. In any case the total cost is:

$$c_{two,left} = 2 \cdot f_w + 12\sqrt{3} \cdot t \cdot c \qquad c_{two,right} = 2 \cdot f_w + 18 \cdot t \cdot c \qquad [20]$$

With $12\sqrt{3} \approx 20.8$ it is clear that the right hand side of figure 18 is the optimal solution. But please note that this result is due to the chosen geometry. Therefore, the second equation of [20] gives the true result for c_{two}. Equating [19] with [20] gives the chaotic regime (transition from one warehouse to two) at:

$$c = \frac{\sqrt{3}}{18} \cdot \frac{f_w}{t} \qquad [\ 21\ ]$$

With further increasing values for c, three or four warehouses should become possible. The optimal placements of these warehouses are always at a customer side. The corresponding costs are easily calculated by:

$$c_{three} = 3 \cdot f_w + 6\sqrt{3} \cdot t \cdot c \qquad c_{four} = 4 \cdot f_w \qquad [\ 22\ ]$$

To see it more clearly it is very useful to plot the total costs for one, two, three, or four warehouses:

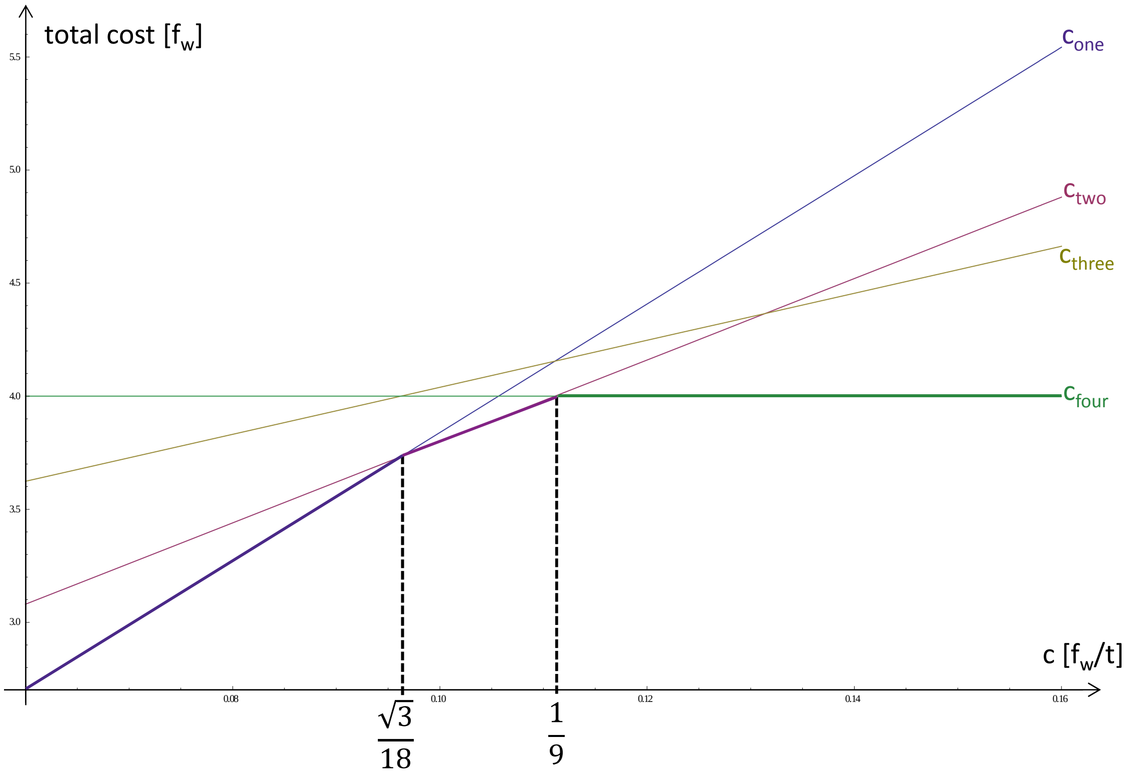

Figure 19: Total cost for different number of warehouses and four customers

With any number of warehouses the total cost increase with consumption c. But at $c = (\sqrt{3}/18)(f_w/t)$ there is a jump from one warehouse to two. Then at c = (1/9)(f_w/t) there is a second jump from two warehouses to four. (Having three warehouses is never optimal.) So the two dashed lines in figure 19 indicate the chaotic regime here. Here they are just points in the one dimensional c-space. If the c_i were different, they were three dimensional objects in the four dimensional c_i-space.

Figure 19 is important for a quite different reason. As I have shown in detail, the optimal warehouse position may behave chaotically. Therefore it cannot be predicted. This is in accordance with statement 1. The position of a warehouse is not a conserved quantity. If a position of one warehouse changes, there is no necessity that anything else changes. Therefore, it comes as no surprise that the warehouse positions behave chaotically under sufficiently complex situations such as finding optimal warehouses. The immediate question is here: Are there conserved quantities? They must not behave chaotically. Indeed, the total cost is a conserved quantity. It cannot be changed without any other corresponding changes. As one sees from figure 19 the cost is a perfectly fine function of customer demand c. Even at the chaotic points indicated by the dashed lines it is continuous. The change in cost is arbitrarily small for any arbitrarily small change in consumption c.

To close this section I will comment on a situation with any number of customers. In the same manner, one can go ahead with more and more customers. However, it will become more and more tedious each time. But it is also possible to make the most general approach of an arbitrary number of customers. (But it is not possible to solve it) This is done in the appendix 3.1. There a customer density distribution field c(x,y) is assumed and defined in [68]. Please note that the arrangement of distinct customers is also within this approach. The setup of, for example, three separate customers is easily achieved by a density distribution of three delta distributions (for a definition of delta see [69]; the details of the limit are given in appendix 3).

The general procedure is as simple as with a handful of customers. One has to calculate the total cost for one warehouse and for two warehouses, and equate these two costs. The severest difficulty is to decide which warehouse is supposed to serve which customer. It can be determined by solving a non-linear partial differential equation. Even numerically this is a great challenge. Formally one obtains [23] by equating the total costs (for details please see appendix 3.1):

$$\iint_{-\infty}^{\infty} dxdy\, c(x,y)\cdot\sqrt{(x-w_x)^2+\left(y-w_y\right)^2} = f_w + \int_{-\infty}^{\infty} dx\, \theta(y(x)-x)\cdot\left[I_1\big(x,y(x)\big)-I_1(x,-\infty)\right] + \int_{-\infty}^{\infty} dx\, \theta\big(x-y(x)\big)\cdot\left[I_2(x,\infty)-I_2\big(x,y(x)\big)\right] \qquad [\,23\,]$$

The I's in [23] are essentially integrals of the customer density c(x,y). The function y(x) is a functional of the customer density c(x,y) and a solution of the above mentioned non-linear partial differential equation. Though solving [23] for c(x,y) is *extremely* complicated (even numerically), there *are* solutions. More precisely, the functions c(x,y)fulfilling [23] are building an infinite dimensional function space. For all elements of this function space the warehouse distribution is totally chaotic. Even the smallest variations in the customer distribution cause a change from one warehouse to two or vice versa.

2. Learning curves

Rapitio mater sapientia est, says an old Latin proverb. And of course, due to repetition one learns to avoid mistakes. And one will be able to do something quicker, better, or in the language of business with less cost. Because planning cost (or revenue) is the key managerial task, planning learning is essential. The archetype of learning may take place in production and similar operational tasks. However, it is by no means limited to it. Starting with a new product one has to learn its marketing, for example So, it comes as no surprise that some of the oldest formulas of management science are the formulas of learning curves. So far this has to do nothing with the topic of this thesis. However, planning with learning curves starts with the present progress like:

week one: cost were 15.00 € per 100 pieces
week two: cost were 12.00 € per 100 pieces
week three: cost were 10.00 € per 100 pieces

The interesting question is how this progress will go on, and how many parts per hour will we produce in the long run? Having a formula for cost decrease due to learning, one will try to fit this formula with the cost data like above. In doing so one is able to predict the future. More general, there are input data (present cost data here or consumption rates in the previous section) from which output date are calculated (future costs here or warehouse positions in the previous section). Needless to say, if the input data vary, so will the output data. So there is a possibility that for an extremely small change of input data a dramatic change of output data might happen. In other words there is chaos which makes any planning totally useless. Exactly this is the connection with this thesis:

- Are there chaos effects in the application of learning curves?
- What can be done in chaotic situations? Are there conserved quantities?

The answer to the first question is “yes.” The second question can be answered within particular examples.

It sounds almost surprising that such chaos effects in learning curves hasn’t been discovered long ago especially because learning curves are old and frequently used by production managers which are “fluent” in mathematics. However, traditional learning curves do not show chaos effects, but they are wrong. The first publication of mathematically correct learning curves can be found in Grabinski (2007) and its generalization in Klinkova (2012). Therefore, it is useful to show the general idea behind correct learning curves in the next subsection. Then I will apply it to an example of marketing where normally two parties learn to fight for market share.

2.1 Learning curves for one and two party learning

Traditionally learning curves are assumed to take the form

$$cost = c_\alpha \cdot t^{-\alpha} \qquad [\ 24\ ]$$

So the cost decreases over time. The positive exponent α is typically around ½. The pre-factor c_α has some strange dimension: cost times time$^\alpha$. [24] or a similar version can be found in many books and lecture scripts about management science. Quite recently it had been applied to the mutual "learning" in warfare in the prestigious journal Science, Johnson (2011). However, [24] is fundamentally wrong. And it can be shown very easily. Just take the limit $t \to 0$ and $t \to \infty$, respectively in [24]. It leads to cost $\to \infty$ and cost $\to 0$, respectively. And of course cost is neither infinity in the beginning nor is it zero in the end. (One may add some constant c_∞ in [24] which makes the cost different from zero in the long run. In the same way one might introduce a time t_0 by substitution $t \to t_0 + t$. but this would introduce a fundamental time t_0. In other words, if our world wouldn't be homogenous in time, statement 2 wouldn't hold. But this bares any experience so far.) So it comes as little surprise that standard textbooks or lecture note lack a rigorous proof of [24]. The first who has realized it was Grabinski (2007).

Before discussing the correct form of learning curves I will briefly comment on the possible derivation of [24]. As stated, management science says little on it. However, Johnson (2011) did explain where his approach came from. He derived it from a random walk approach as stated in, for example, Rudnick (2010). In a random walk one considers, for example, a person who steps with say 50 % probability to the right and to the left. On *average* he doesn't move. For $t \to \infty$ one will be at the origin. And the probability being, for example, three steps to the right decreases with time in a power law as in [24]. Trying randomly will lead to either wrong steps or right ones. Of course there are more advanced random walk approaches where there are steps in many different directions. All such procedures show the power law behavior like [24]. So if one assumes that learning is just a random walk, then one must have a power law behavior. However, this kind of learning would be the learning of non-thinking creatures like ants. And indeed, for ants "learning" to find the right place takes exactly the form of [24]. However, more advanced animals and (hopefully) human beings learn differently. They can think. They analyze what went wrong after each step, and try to find the mistake to be avoided in the next step. In this sense, learning (in production, etc.) means finding a mistake to be avoided next time. In the beginning there are lots of mistakes. To spot one of them is simple. And one will probably find the one most easy to spot. With each mistake discovered, there are fewer mistakes left which are harder and harder to spot. The rate (timely derivative) of mistake spotting will be inversely to the number of mistakes discovered already. Grabinski (2007) translated this approach into a differential equation. Its solution yields:

$$cost = c_\infty + (c_0 - c_\infty) \cdot e^{-t/\tau} \qquad [25]$$

c_∞ is the cost after all mistakes had been discovered. It is the lowest possible cost after finding the optimal way. And c_0 is the cost in the beginning while trying for the first time. τ is the time scale on which the learning takes place. Many useful applications of [25] can be found in Grabinski (2007). But they are beyond the topic of this thesis.

In summary, [24] is the learning curve for unconscious (random) learning as it is present at simple animals like ants. In contrast, [25] describes conscious learning as done by highly developed creature like many apes or humans. So, it could be an ideal tool for behavioral biologists which want to decide whether an animal *thinks* or not. They could just fit [24] or [25], respectively with the corresponding learning data and the better fit could give a statistical proof of consciousness. Unfortunately the graphs of the functions in [24] and [25] look pretty similar if the constants are properly chosen:

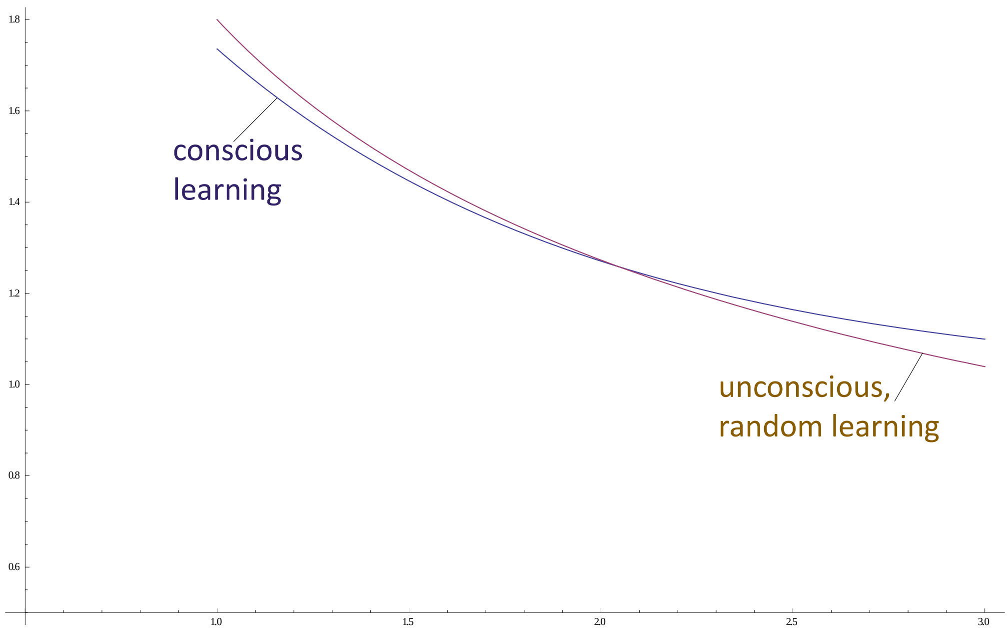

Figure 20: Learning curves for conscious and unconscious learning

This may also explain why the wrong use of the “learning curve” of [24] has not been discovered in many operational situations where it had been used quite successfully.

Before I close this section, I will come to an important generalization of learning curves. Normally people think that one person is learning how to perform a task and not vice versa. For example, a person learns how to drill a hole in less and less time, but the processed material does not learn to increase the time. Logically, this kind of learning is not applicable to most business situations where competition is present. There are (at least) two parties who learn how to compete with each other. This two party learning has been considered only recently in Klinkova (2012). There the problem of two parties fighting against each other has been considered. It was a direct response to the work of Johnson (2011). Deriving the learning curve for two party learning is a simple generalization of the derivation of [25]. Instead of starting with one differential equation, one has two coupled one. The details can be found in Klinkova (2012). The general form of such a learning curve takes the following form:

$$A + B \cdot e^{-\frac{t}{\tau_b}} - C \cdot e^{-\frac{t}{\tau_c}} \qquad [\ 26\ ]$$

There is no equal sign in [26] because I have left it open *what* is learnt here. Normally it is not cost. In order to explain it I will take an example which is also used in the next subsection. One may think of the typical situation of competition in the product (or service) market. While introducing a new product, one has to place it in the market. One will learn the particularities of marketing. This means one has to avoid the mistakes. But there are also competitors fighting for revenue. Both sides will learn. Normally one knows a lot about oneself, but not so much about the competitors. One easily sees an in- or decrease in revenue. But one does not know where it leads to eventually and one does not know whether it is due to one's own fast learning (good) or weak and slow learning competitors (may be dangerous). To answer these questions is the purpose of a two party learning curve as the one in [26]. To see the point one should assume that [26] describes the change in revenue of a (new) product. In the beginning (t = 0) the revenue is A + B – C. Over the time one will lose revenue to the competitors due to their learning, but will also get revenue from the competitor due to one's own learning. In the end ($t \rightarrow \infty$) one's revenue will be A. Obviously one lost B (to the competitors) and one gained C (due to one's own learning). Therefore, τ_b is a measure for the learning speed of the competitors and τ_c is the learning speed of oneself. Of course, the constants A, B, C, τ_b, and τ_c are generally not known, except one has a waste of experience in similar situations. In general one has to observe market data (here simply one's own revenue) for a while and then fit the data to [26] in order to get the desired constants A, B, C, τ_b, and τ_c.

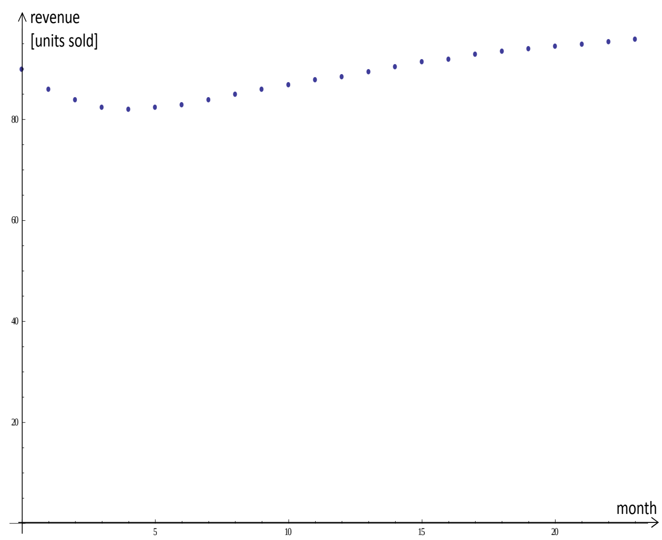

Figure 21: Revenue with new product per month

In order to see the point more clearly and in preparation of the next chapter one may consider a company having an undisclosed amount of competitors starting with a new product. Every month they measure the number of units sold, for example zeroth month 90 units, first month 86 units, second month 84, and so forth. The entire revenue data for 24 months are given in figure 21. These data must be fitted with the function of [26]. It can be done by an ordinary least square fit. Please note that some constants in [26] are in the exponent. Therefore, the least square fit leads to five non-linear equations which can be solved numerically only. Though this is straight forward, the particular numerics is partly tedious and slowly converging, which is the mathematical reason for the chaos effects discussed in the next subsection. Be it as it may, the least square fit leads to

$$A = 100.3 \frac{\text{units}}{\text{month}}, \quad B = 25.92 \frac{\text{units}}{\text{month}}, \quad C = 36.13 \frac{\text{units}}{\text{month}}, \quad \tau_b = 3.062 \text{ months}, \quad \tau_c = 10.87 \text{ month}$$

With it the function of [26] is well defined, and one may plot its graph:

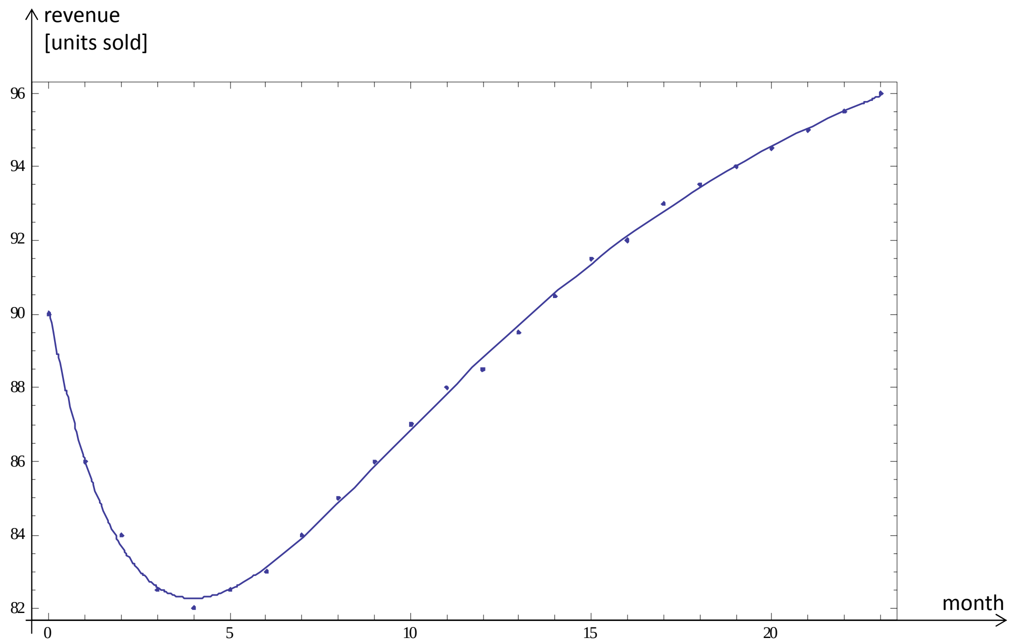

Figure 22: Revenue per month actual (dots) and predicted by fit (line)

Compared to figure 21, the vertical scale of figure 22 has been enlarged in order to see the behavior more clearly. (The data are of course the same.) Interpreting this result is most interesting for a marketing analysis. Firstly one can predict one's own revenue in the long run. It will be A = 100.3 units per month. It will be the revenue if everybody in the market has learnt to avoid all mistakes. Because $\tau_b \approx 3$ months (learning period of competitors) is much smaller then $\tau_c \approx 11$ months (one's own learning period) the competitors are learning much faster. They are learning in three months what one learns in eleven months. As an advice one can say that this company has a good product but is very slow in bringing it to the market, especially if compared to the competitor. Looking at figure 22 one sees that it takes 20 months to reach 95 units per month. Extending the plot yields 50 months for 100 units per month. Depending on the cost situation that may be too long to survive. All this and maybe much more useful information can be drawn from this simple least square fit. The only necessary input data were one's own revenue. The most exciting result is to get an estimate for the learning speed of the competitors without knowing anything of them even not how many units they are selling.

But because this thesis is not about marketing, I will stop the discussion at this point. I will jump to the next subsection where I show that this has to do a lot with chaos effects and conserved quantities.

2.2 Learning and chaos effects in marketing

In the last subsection I have shown how to get useful information for product marketing from applying (two parties) learning curves. Because the results are numerical results, chaos effects or chaotic regimes cannot be calculated. But they can be shown on statistical bases. I have taken the input data from figure 21 (revenue per month) and made them fluctuate within a Gaussian distribution. I have chosen a very narrow distribution with a standard deviation σ = 0.001 or a variance of σ^2 =10^{-6}. With such a narrow fluctuation the results (our constants A, B, C, τ_b, and τ_c) should also fluctuate on a narrowly for non-chaotic behavior and widely for chaotic behavior.

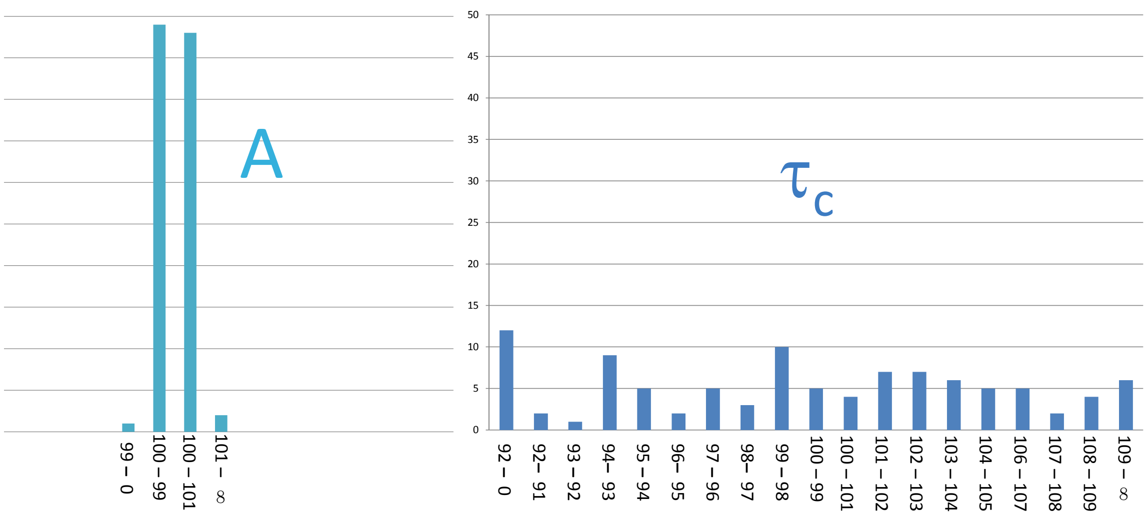

Figure 23: Distribution of A and τ_c, class width percentage from average

For the two constants A and τ_c this is displayed in figure 23. In total, I have calculated the constants for 100 different (Gaussian) distributions of the input parameters. (For the detailed result please see appendix 4.) In figure 23 I show the distribution of the hundred output values for A and τ_c within percentage classes. So the class "100 – 101" means for example the number of results within 100 % and 101 % of the average. As one sees, "A" shows an extremely narrow distribution. It does not behave chaotically. However, τ_c shows an extremely wide distribution. It is scattered around (almost) randomly. This is one of the standard definitions of chaos as found in textbooks such as Schuster (1984). With the slightest change in input parameters of a percent or less the value of τ_c is scattered around randomly in the $\pm$ 10 % range. Besides this direct proof of chaos effects in a business situation, there are two important interpretations of this result.

The first surprise is that τ_c behaves completely chaotic while the constant "A" is perfectly regular. This is by no means coincidence. "A" must not behave chaotically. Therefore it was predictable. (Actually this is the reason why I investigated in this problem in the first place. I wanted to show that my statement 1 holds.) "A" is the revenue the company makes in the long run with its new product. And revenue is a perfectly conserved quantity. Making revenue means someone is paying for it. So it cannot change without a corresponding change somewhere else. It is the archetype of a conserved quantity. This is in accordance with statement 2, which states that value is the conserved quantity in business and economics. Of course, value is not equal to revenue, but the value of this new product is a derivative of the revenue it will create. Please note that one may make revenue with the selling of collectors' items like the "Blue Mauritius." Such revenue is connected to a so called value which is actually the market price Appel (2011). In any case, revenues like the one by selling collectors' items are never described by the model of [26]used here. Therefore this causes no contradiction.

As one sees, the conserved quantity *revenue* is very predictable. The dynamics of the system (given by the other constants of [26]) cannot be predicted due to chaos. In other words, the outcome is predictable, but the way and speed is not. This is completely analog to the weather forecast. The precise date when it will rain is unpredictable in the long run. The total amount of rain is very easy to predict due to mass conservation (of water in this case).

The second important question is whether the system is chaotic or the underlying model. This important question was asked for the first time by Grabinski (2004). Obviously I have undoubtedly shown that for example τ_c is unpredictable due to chaos. However, for sure I have shown that [26] shows chaos under the circumstances discussed here. I have *assumed* that the revenue given in [26] describes the true development of revenue. There are good arguments that it does, as discussed above. However, all this is no proof. In any case it is a model and every model has its limitations. Maybe these chaos effects just show us the limit. To decide on this is next to impossible. If a conserved quantity would show chaotic behavior, then it would be sure, that the model does not describe the reality under the given circumstances. Actually, this is the main point for the next section. In the "diffusion model" of marketing 20 years ago chaos has been "discovered." However, there chaos also occurred in conserved quantities. Therefore, there must be some faulty logic somewhere on the way.

To find out whether reality is chaotic or the underlying model is very difficult in general. Science can use experiments to see whether the world behaves chaotically. In exactly that way it was possible to show, that the development of the weather is chaotic in the long run. Edward Lorenz could just show that his system of equations behaved chaotically. And he assumed that they described nature correctly. However later experiments on systems like air streams showed that there is chaos present. In business and economics "experiments" like the one in science are impossible. The only chance is to consider real situation over many years. From time to time there are situations where the initial conditions are almost identical. If the outcome in these situations is very different that would mean chaos. However, if two situations are many years apart it is mostly impossible to speak of identical initial situations because the environment (for example technology) has changed so much. A possible playground where business is fast and repeat almost identical is the stock market. While the stock price is not a conserved quantity (Appel (2011)), it may behave chaotically and almost all observations show that it does. Therefore one finds spurious correlations between the weather and the stock prices, for example Hirshleifer (2001). Though this correlation may be less spurious than one thinks, cf. Akerlof, Shiller (2009).

3. Diffusion model in marketing

The diffusion model is a tool in marketing applied in order to estimate how a (new) product will find its way in the market. This thesis is far from being a thesis about (quantitative) marketing. But this model is very helpful to understand the mechanism of chaos and conserved quantities. Moreover, it was Weiber (1993) who stated the end of ordinary diffusion modeling due to chaos. As mentioned earlier, this was is one of the very few reasonable attempts to look for chaos in business and economics. (Mostly considerations of chaos had a more or less esoteric character.) However, the work of Weiber (1993) contains some (mathematical) mistakes or misunderstandings. Correcting these mistakes eliminates the seemingly chaos effects as described by Weiber (1993). Applying the conservation laws derived here, it would have been clear even without considering any mathematical details.

Before going directly to the work of Weiber (1993), I will briefly explain the background of the diffusion model. If the first piece of a new product is pushed into the market, it has almost no market share. Then potential customers (and maybe competitors) are learning about the product. They want to have it too. This effect becomes enhanced if more people have the product. Of course, there is something like market saturation either due to the fact that all potential customers are using the product or a competitor fights back. In any case one expects an equilibrium state. Of course, the marketing manager wants to predict how big the market share is in this equilibrium state and how fast it is reached. This is quite analogous to the diffusion in physics. Heat diffuses from the warm end to the cold one leading to a homogeneous temperature in the end (equilibrium state). Closer to marketing is the diffusion of ozone emitted in a closed room by an electrical machine. There may be a ventilation due to an open window (corresponds to competitor). Eventually there will be a stationary state where the rate of ozone emitted is equal to the amount of ozone going out the window. The question is when this equilibrium is reached and how high the concentration of ozone is in the room. Safety engineers might calculate it by using diffusion equations from physics.

From all this marketing research postulates three different models. The first is called “exponential diffusion model:”

$$N_t = a \cdot (M - N_{t-1}) \qquad [\,27\,]$$

N denotes the number of customers purchasing the product. In the beginning there are N_0 customers, then N_1 and so forth. M denotes the saturation of the market and “a” is some diffusion coefficient which determines the speed how fast the saturation M is reached. In this model the number of new customers N_t is proportional to the distance to market saturation.

In a second model one assumes that the gain in new customers is (simultaneously) proportional to the distance to market saturation *and* the number of existing customers. It leads to a product of N and M – N, or the so called logistic diffusion model

$$N_t = b \cdot N_{t-1} \cdot (M - N_{t-1}) \qquad [\ 28\ ]$$

Here "b" is a constant analogous to "a" in [27]. While [27] and [28] describe some form of diffusion, a combination of both may be considered most general:

$$N_t = a \cdot (M - N_{t-1}) + b \cdot N_{t-1} \cdot (M - N_{t-1}) \qquad [\ 29\ ]$$

For whatever reason, this model is called semi logistic diffusion model. Weiber (1993) sticks to the logistic diffusion model in his work. Furthermore he sets M = 1. This is by no means a limitation. It just states that one sets the market share reached eventually to 100 % or 1. (That does not mean that one will reach 100 % market share.) However, [28] with M = 1 is identical to [4] or the definition of the logistic map. So it comes as no surprise that Weiber (1993) finds chaotic behavior starting from "b" around 3.6. (From this it is clear why Weiber (1993) did not consider the exponential diffusion model and semi logistic diffusion model. The first one does not show chaos, and the second one shows the same chaos effects as the logistic diffusion model.)

The interpretation of such chaos effects is difficult. Is it really so that the market share increases and decreases rapidly for high enough values of "b?" Why is it the case? And why does it start quite suddenly at "b" around 3.6? Or does the diffusion model break down at this point and is not valid anymore? In contrast, when deriving the diffusion model there is no limit in validation. Mathematically spoken this would prove that the diffusion model is not valid at all. (It is enough to show one counter example to falsify a theory!) It becomes even worse if one knows that the logistic map is defined only for starting values (here N_0) between zero and one, and a constant (here b) between one and four. Else it might diverge. If the diffusion model for marketing is a logistic map, why there are such limitations? In the real world there should not be such a limitation. Therefore, a "b" bigger than 4 should also be possible. However, b > 4 will also produce negative market shares.

All this does not make sense at all. However, the confusion has nothing to do with chaos or even the entire diffusion approach in marketing. Diffusion does make sense statistically only. Be it in physics or marketing, there is a single contact which leads to a new state. In physics it may be the scattering of two particles and in marketing it may be the contact of a new customer with a potential one which may or may not convince the new customer. Each of these contacts is very different and needs a complete individual consideration in order to calculate the outcome. Considering the model of, for example, [28] two things must be taken into account:

- The *constant* "b" is different within <u>every</u> period t.
- Each period might have a different length.

So, there are two constants (length of period and “b”) for every period. Considering, for example, 20 periods would mean to have 40 unknowns. This would make the entire model correct but useless. Please note that Weiber (1993) says *something* about a time dependent “b,” but not about a different period length. Actually he uses a g_t instead of “b” in [28] in the first place. Then he assumes that the most general form of $g_t = a + b\ N_{t-1}$. With this faulty assumption he comes up with [27], [28], and [29]. This is by no means rigorous. In order to overcome the two problems above one may consider a statistical average only. This means that each contact (like each scattering in physics) is completely different. However, nobody wants to know the outcome from each contact. One assumes an average outcome at each contact. On a time scale large compared to a single contact or scattering this will give the true result. In doing so, one can calculate the average of many incidents. In this sense [27], [28], and [29] are nothing but differential equations. They are easily solved by separation of variables and integration. Doing so leads to a function t = t(N) and the reverse function is the desired function N = N(t). Please note that the results are straight forward but pretty clumsy. (For a detailed discussion of the mathematics please see appendix 5) Furthermore, the mentioned reverse function is normally not a combination of standard functions. However, setting M = 1 leads to quite simple expressions. (As mentioned this is not a simplification: it is just a setting of a particular scale for the market share.) The exponential diffusion model yields:

$$N(t) = e^{-(a+1)t} \cdot \frac{(a+1)\cdot \mathrm{N}_0 + a \cdot e^{(a+1)t} - a}{a+1} \qquad [\ 30\ ]$$

Solving the logistic diffusion model yields:

$$N(t) = \frac{(b-1)\cdot \mathrm{N}_0 \cdot e^{bt}}{b \cdot \mathrm{N}_0 \cdot e^{bt} - e^{t}[b \cdot (\mathrm{N}_0 - 1) + 1]} \qquad [\ 31\ ]$$

Please note that the solution of the semi logistic model is not the sum of the two above. Because it is slightly more complex it is not displayed here but in appendix 5. Both equations ([30] and [31]) show no chaotic behavior for any values of “a” or “b.” One may also choose any starting value and “b” might take values over four, which would lead to divergence in the logistic map. So the chaos of the logistic diffusion model vanishes without a trace. Because N is the gain in market share, N is a conserved quantity (gaining market share means losing some for somebody else), it must not behave chaotically. Please note that one may find chaos in quite some other way here. If someone takes the logistic diffusion model ([31]) and observes the real market data for some while, one may use it as a fit for “b” of [31]. This is analogous to the last section where I have fitted [26] with marketing data. One might expect that this will also lead to a chaotic behavior of “b.” However, as one sees from some other consideration below, “b” has a lot to do with the eventual N (or $N(t \to \infty)$), it must not behave chaotic.

It is another question why [27], [28], and [29] have been used successfully in marketing research for decades, though [30] and [31] are the correct ones. The answer is pretty simply. Consider for instance [28] (logistic map). It is a non-linear differential equation with the solution in [31]. Rather than solving a differential equation exactly one may use various approximations. One way is to go by steps. Such iteration is always possible in linear differential equations. In nonlinear differential equations it may or may not work. And exactly this is the case here. For small "b" it works and for bigger "b's" it doesn't. In order to see the point more clearly I have plotted N(t) of [31] and N_t of [28]:

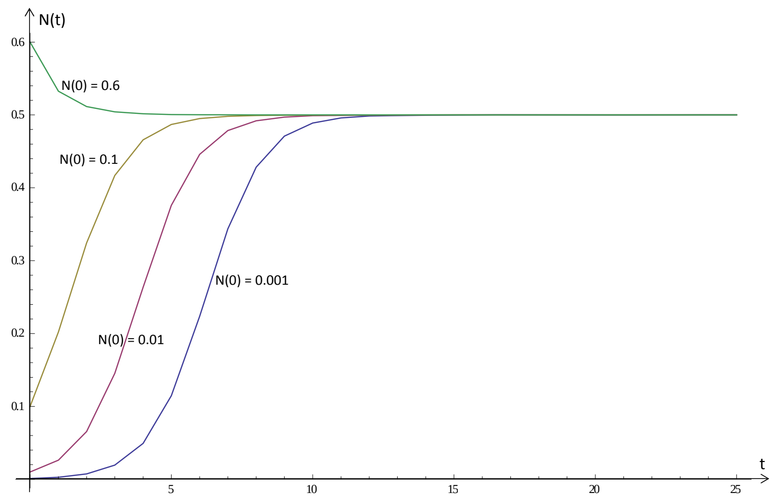

Figure 24: Plot of [31] (b = 2) for four different N(0)

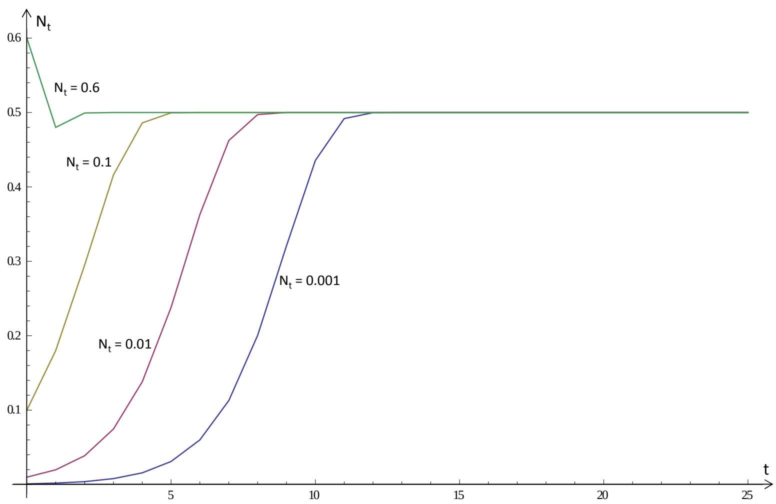

Figure 25: Plot of [28] (b = 2, M = 1) and four different starting values N_0

In figure 24 one sees the exact solution of the logistic diffusion model (b =2) and in figure 25 its iteration. As one sees, it is a quite good approximation. In both plots I have chosen a discrete plot for t =1, 2, …, 25 in order to make it comparable. The same can be done for b = 4:

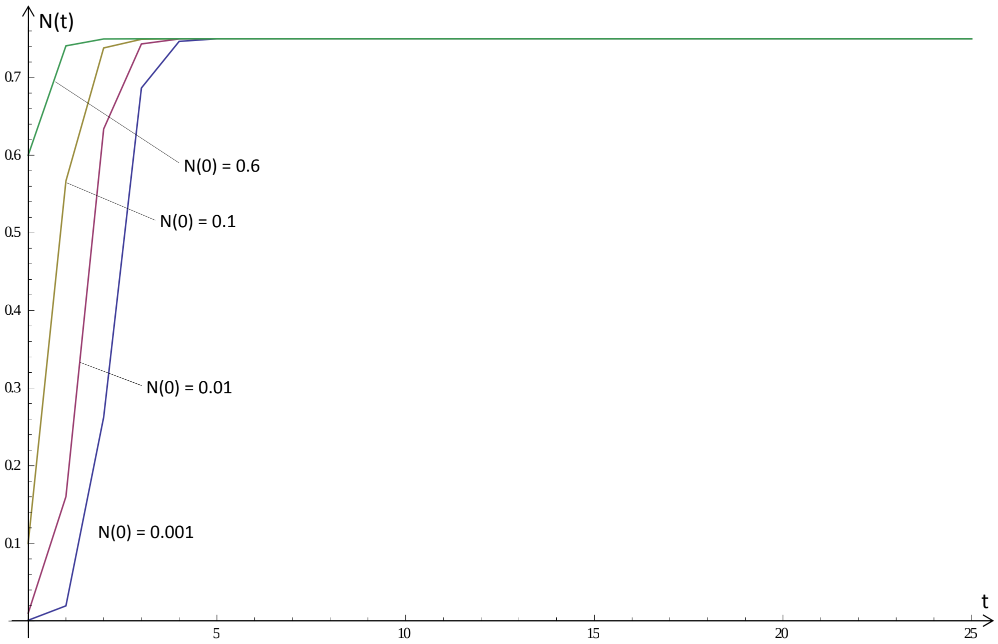

Figure 26: Plot of [31] (b = 4) for four different start values N(0)

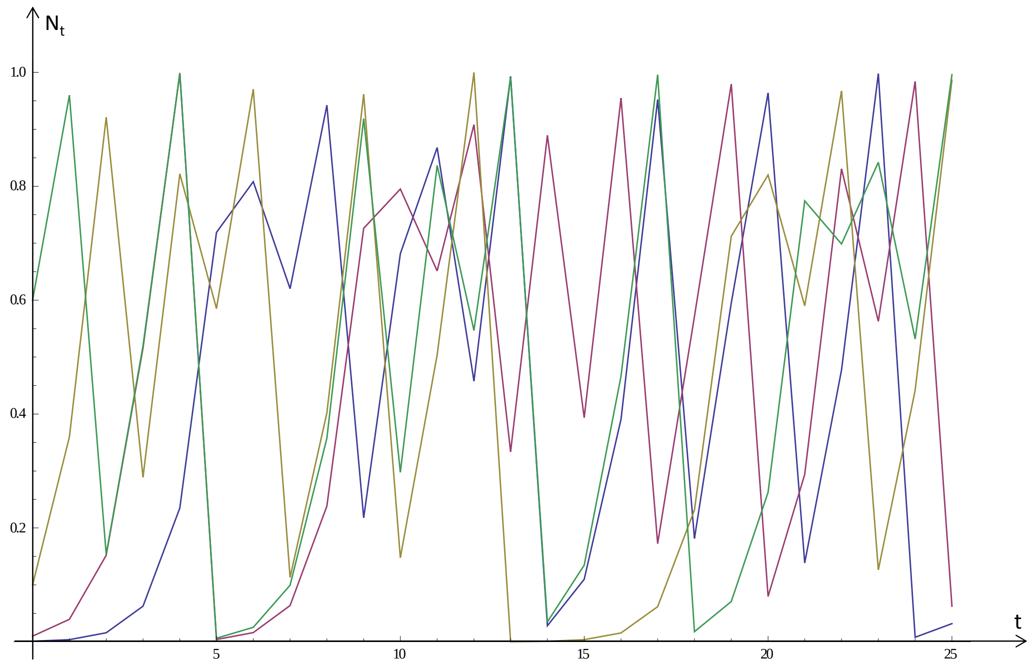

Figure 27: Plot of [28] (b = 4, M = 1) for four different start values N_0

Here the iteration of figure 27 looks completely different from the exact solution of figure 26. The exact solution looks pretty regular as expected. Its iteration shows a complete chaotic behavior. It is obviously a pretty poor approximation.

To summarize, the chaos in the diffusion model as announced by Weiber (1993) is nothing but a faulty solution of the diffusion model. Because the diffusion model gives nothing but the market share, it cannot yield chaotic results. This is because the market share is a conserved quantity. (If someone's market share increase, somebody else's must decrease correspondingly.) However one could also use the diffusion model differently, though up to my knowledge nobody ever did so. Take, for example, the solution of the logistic diffusion model [31]. Here one may take N_0 and "b" as fit parameters. N_0 has the dimension of market share and cannot vary chaotically. 1/b is something like the "time to market." It can vary chaotically. Please note that this use of the diffusion model is something very similar to the procedure of the last section (two party learning curves).

4. Development of chaos in financial markets

The most interesting area of planning and forecast is probably the financial market. Most people are dreaming of knowing tomorrow's stock prices. The interest is similar like knowing next week's lottery numbers. And indeed it is similar in many respects. Firstly, having a (scientific) procedures to forecast either lottery numbers or stock prices is of no use whatsoever. Simply because everybody would know it and nobody could make a profit. (Knowing the numbers or prices in a fraudulent way would be an advantage.) Secondly, lottery numbers and stock prices are unpredictable due to chaos. In the case of lottery numbers it is very easy to see. In principle drawing lottery numbers is totally deterministic. The laws of classical mechanics describe it perfectly. However, it is pretty complex, so that the gravitational effects of people standing in the near surroundings are very important. This is like the weather forecast and the butterfly wing effect. Everything is deterministic but due to chaos effects it appears to be random. That this is the case in the financial market too, has been stated by Grabinski (2004) already. Later it was scrutinized by Appel (2011) more rigorously. He also used conserved quantities to give a reasonable description. This section will not build on it or draw further conclusions (which are important). The purpose here is to show how chaos develops in financial markets or why other markets behave non-chaotic.

Summarized in a new book (Bogle (2012)) there is a distinction between "investment" and "speculation" in stocks. (Though already many years old it is essentially ignored.) It was Appel (2011) who took it more quantitatively:

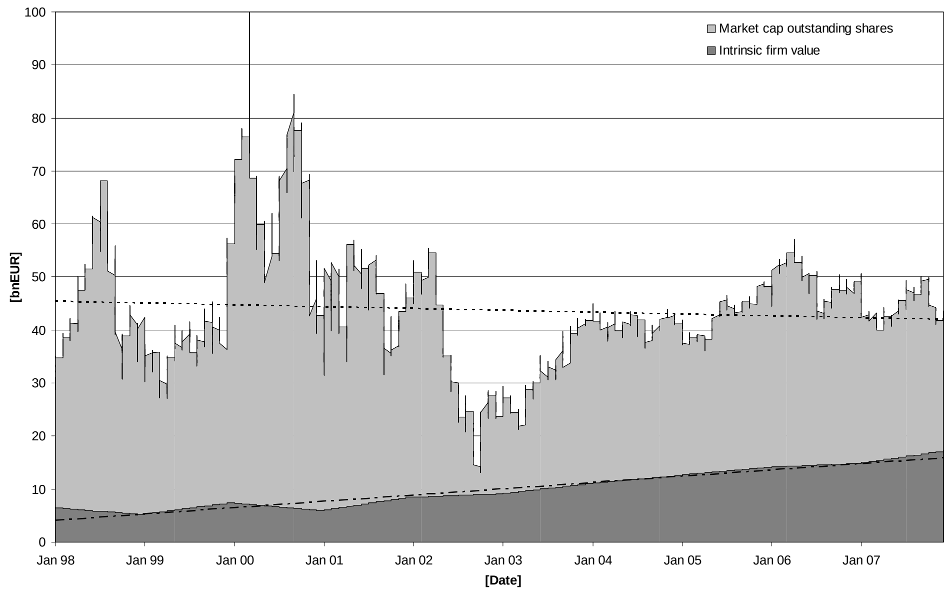

Figure 28: Intrinsic and speculative value of SAP (source: Appel (2011))

In almost all cases the stock price is far higher than the current profit of a company suggests. However, this is no argument for investments bankers and the like. They say the current price reflects the future earnings, which is reasonable. Because nobody knows the future, they are always right. In figure 28 the well-known stock price of SAP for a period of 1998 till 2007 is displayed (lightly grey area). It is fluctuating rapidly. However, for this area the future is (essentially) known. Knowing the earning of the future one can calculate today's value. This is done in the dark grey area in figure 28. It is the intrinsic and conserved value of SAP. Because SAP was growing and made god profits, it is a monotonous increasing function. Because nothing extraordinary happened to SAP, it is a very smooth function in contrast to the lightly grey area. It reflects the speculative part. It looks as varying chaotically. In what follows in this section I will show that it really does and why.

As a first step I will show the behavior of "normal" markets. They are typically displayed by supply and demand curves leading to a market price and a traded volume:

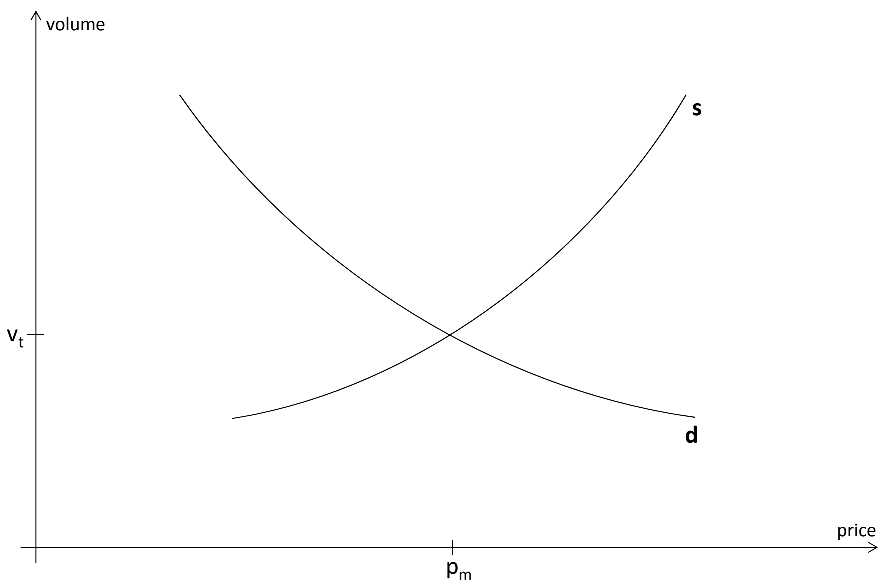

Figure 29: Supply and demand of "ordinary" market

If the price goes up less people are willing to buy (curve d). The opposite is true for the people who want to sell (curve s). The intersection of both curves gives the market price p_m and the traded volume v_t. One major ingredient is that the demand curve is monotonously decreasing and supply curve is monotonously increasing. It will guarantee an intersection. Please note that the curves in figure 29 do not mean that a certain amount of people are buying or selling at particular prices. At a price p a certain amount of people are willing to sell or buy. The corresponding volume v is the accumulated volume being traded by also this people. If there is a market price, of course, everybody is buying and selling at that price. The positive or negative slope in the supply or demand curve seems to be the only reasonable possibility. However, that higher prices can attract more buyers, was already used in order to explain the momentum effect in stock markets, Appel (2012). It is also used by Schefczyk (2012) to explain the wrong effects in financial markets.

Or one can look at figure 28. Nobody paid the stock price of SAP because he or she believed in a future cash flow several times higher than it actually became. Most investors saw a rising price and "speculated" that it will increase further. If demand and supply curves are positively sloped, there is quite often no market price:

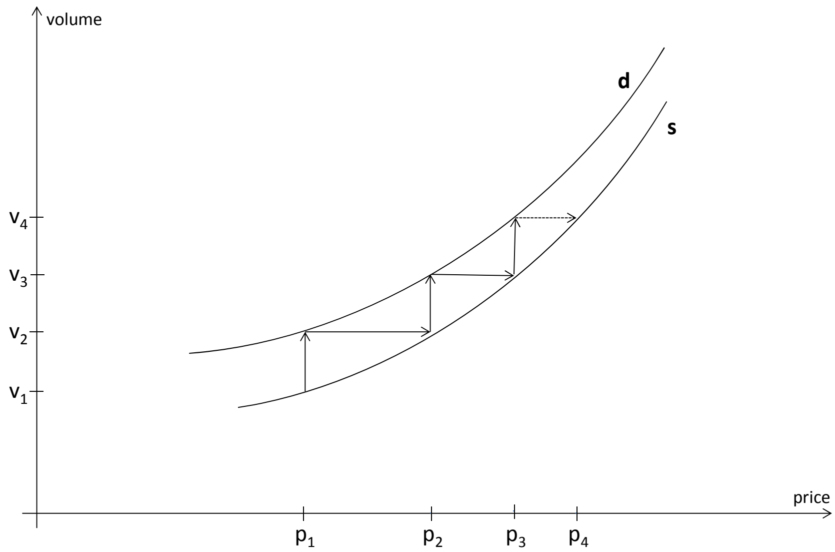

Figure 30: Supply and demand in typical financial market

Starting with p_1 in figure 30 there is too little supply or too much demand. Therefore the price will go up to p_2. But the volume v_2 is still too small. So the price will go to p_3 and so forth. It will spiral upward. If one exchanges s- and d-curve in figure 30 the price will spiral downward. This is a good indicator for chaotic behavior, but it is no proof for instability. Nevertheless, there may be points where supply meets demand. One may also consider a more complex demand curve:

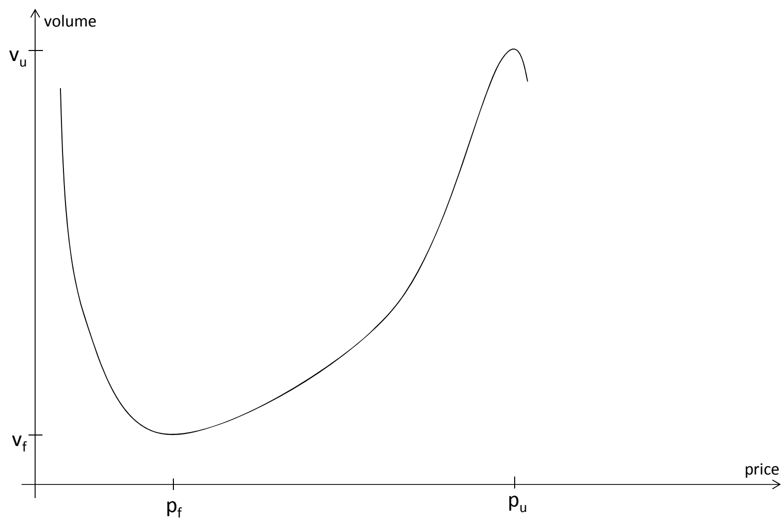

Figure 31: Typical real demand curve for financial asset

Between p_f and p_u is the typical behavior for an financial asset in figure 31. The increasing price makes the paper more interesting. This is quite analogous like for luxury goods which are bought in order to show that the buyer is rich. Therefore, it is more "valuable" for the buyer if it is expensive. Under a certain threshold (here p_f) it becomes an ordinary object. The stock has a price of around its discounted cash flow or the (luxury) watch has a price comparable to an ordinary one of same quality. Logically demand goes up if the price goes down. As stated for prices higher than p_f the higher price makes the object more attractive. However, the price will not go to infinity. If the price takes an unreasonable value p_u, people will realize the insanity and will not buy any more. Watching this point is very important for luxury goods makers. It is however even more tricky in the financial market. In the jargon of stock brokers it is also called the maidens' hausse. (Because even a maiden will start to invest in the hausse market.) Here no producer is setting the price but the market does. If the considered financial asset is a stock, its volume cannot be arbitrarily big. (Every company has issued a fixed number of stocks.) So, the corresponding volume v_u may or may not be bigger than the total numbers of stocks. So there may be a cut-of before the crash comes. With derivatives it is trickier. They are essentially bets on the future market. And one can make as many bets as one want about the same thing. A hausse market may even encourage issuing more derivatives, and a crash is inevitable.

Up to now I have delivered many arguments why the financial market can be pretty unstable. But this does not prove that it cannot be stable. Of course, there may be also very complex supply curve in figure 31. It most likely intersects the demand curve at some point or points. But the *stability* of a market has to do with the stability of the supply and demand curve. These curves may change in itself and they may do it chaotically. To see the point I will firstly consider the ordinary market of figure 29. Firstly, it should be clear that having a market price p_m is good:

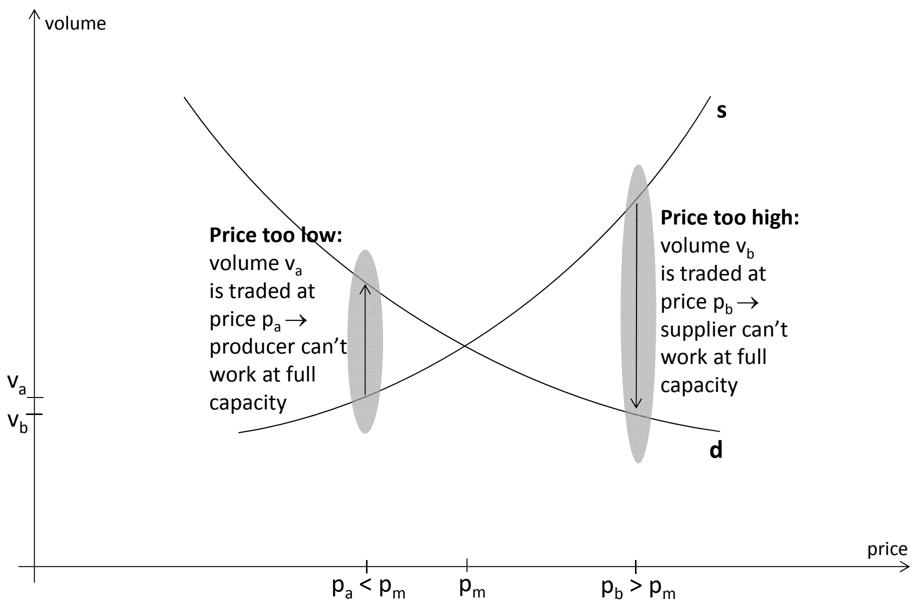

Figure 32: Effect of too low or too high market price in ordinary market

Figure 32 shows that if a trading price is lower than the market price, the traded volume is lower than the optimal volume. As a consequence both suppliers and buyers have less business, and less goods are produced (and consumed) in the end. The same is true for too high prices. Therefore, a *free* market will always turns to a market price like in figure 29. This is also a good explanation why any kind of price setting is always bad. The same is true for other regulations such as import or export taxation or some “buy national” culture. In some sense figure 32 is an alternative explanation for comparative advantage. Unlike the situation of figure 30 people in ordinary markets have an advantage to reach the market value. Furthermore, small variations in any parameter should not lead to a significant change in market prices. In order to see the effect, one can describe figure 29 mathematically. In the closer surrounding of the point (p_m, v_t) the supply and demand curve can be approximated by linear functions of the form f(x) = ax + b:

$$v_{s0} = a_{s0} \cdot p + b_{s0} \qquad and \qquad v_{d0} = a_{d0} \cdot p + b_{d0} \tag{32}$$

The index “0” denotes that this is the market at some starting time t_0. Now the market will vary leading to new constants a_s, b_s, a_d, and b_d. The question is whether this will lead to a (slight) change in p_m or a chaotic variation. In any case, equating the two linear functions of [32] yields the market price p_m:

$$p_{m0} = \frac{b_{d0} - b_{s0}}{a_{s0} - a_{d0}} \tag{33}$$

In an “ordinary” market as discussed here, the supply curve will depend on the cost for the producer. Typically the supplier will sell for say 110 % of its cost. Selling for less than the cost is impossible in the long run, selling for prices much higher than cost will attract competitor which will bring down the price. Similarly, the demand curve is determined by the value for the buyer. A potential buyer will only buy something, if the value for him or her is slightly higher than the sale’s price. A price above value cannot be accepted in the long run, and prices much lower than the buyer’s value will again attract competitors bringing the price up. In other words, the constants a_s, b_s, a_d, and b_d are determined linearly by *cost* or *value*. If cost or value is zero, there is no market and no supply or demand curve. Therefore, each constant of a_s, b_s, a_d, and b_d can be described by a factor times cost or value. Let these factors be α,β, γ, and δ, and [33] can be written as:

$$p_{m0} = \frac{\alpha_0 \cdot \mathrm{value}_0 - \delta_0 \cdot cost_0}{\beta_0 \cdot \mathrm{cost}_0 + \gamma_0 \cdot \mathrm{value}_0} \tag{34}$$

The plus sign in the nominator is due to the fact that the demand curve is negatively sloped and the supply curve positively. In other words, $a_{s0} > 0$ and $a_{d0} < 0$ in [32]. Knowing this leads to positive β_0 and γ_0 in [34]. This makes sure that the nominator will never become zero. Now all quantities with an index “0” in [34] may vary around the indexed values. In other words, [34] is also true if all indexes “0” are skipped. This equation can then be Taylor expanded around the equilibrium values. In lowest order it leads to:

$$p_m - p_{m0} = \frac{\alpha_0\beta_0\text{cost}_0 + \gamma_0\text{cost}_0\delta_0}{(\beta_0\text{cost}_0 + \gamma_0\text{value}_0)^2} \cdot (\text{value} - \text{value}_0) + \frac{\delta_0\text{cost}_0\text{value}_0 - \alpha_0{\text{value}_0}^2}{(\beta_0\text{cost}_0 + \gamma_0\text{value}_0)^2} \cdot (\gamma - \gamma_0) - \frac{\alpha_0\beta_0\text{value}_0}{(\beta_0\text{cost}_0 + \gamma_0\text{value}_0)^2} \cdot (\text{cost} - cost_0) + \frac{\text{value}_0}{\beta_0\text{cost}_0 + \gamma_0\text{value}_0} \cdot (\alpha - \alpha_0) + \frac{\delta_0{\text{cost}_0}^2 - \alpha_0\text{cost}_0\text{value}_0}{(\beta_0\text{cost}_0 + \gamma_0\text{value}_0)^2} \cdot (\beta - \beta_0) - \frac{\text{cost}_0}{\beta_0\text{cost}_0 + \gamma_0\text{value}_0} \cdot (\delta - \delta_0) \quad [\ 35\ ]$$

For any small change, the change in cost and value must be small because they are conserved quantities. Furthermore, α, β, γ, δ will also vary slightly because a competitive market demands that the buyer will buy for slightly less of the value and the seller will sell for slightly more as the cost. Therefore, [35] describes the variation in market price ($p_m - p_{m0}$) perfectly accurate if the environment changes slightly. More mathematically spoken, if all quantities vary with a very narrow Gaussian distribution around their equilibrium, [35] shows that $p_m - p_{m0}$ varies with the same Gaussian distribution. In other words, the market price does not behave chaotically. Proving the stability of an ordinary market involved that there is a competitive market. It would not work with a monopoly. This is pretty clear because a real monopolist may change prices just for fun without any other change in the world.

In the case of a typical financial market (cf. figure 31) the proof of non-chaotic behavior as above does not work. However, that does not prove that there is chaotic behavior. This can be done in another way. Even in financial markets the object being bought has some value for the buyer. This value is normally not conserved. It is normally determined by the future price of the object. The buyer will only buy if a higher price is expected in the future. Therefore, the buyer's value must always be higher than the price. Furthermore, the buyer's value is only slightly higher than the current price, else there aren't enough sellers. (All this is at least true for the typical financial market regime $p_f < p_m < p_u$ in figure 31.) There the following relation must always hold:

$$p_{m0} \lesssim value_0 \quad [\ 36\ ]$$

The value is of course the buyer's (non-conserved) value. Now one may make a small variation leading to $p_m = p_{m0} + \delta p$. Furthermore, the value in [36] is a function of the market price p_m. Therefore, one may make a Taylor expansion in δp in [36]:

$$p_{m0} + \delta p \lesssim value_0 + \frac{\partial value}{\partial p_m}\Big|_{p_{m0}} \cdot \delta p + O(\delta p^2) \quad [\ 37\ ]$$

The immediate learning from [37] is that if the slope $\partial\text{value}/\partial\text{p}_\text{m} > 1$ positive variations in price ($\delta p > 0$) are only possible in the long run. Else the inequality of [37] does not hold. For slope $\partial\text{value}/\partial\text{p}_\text{m} < 1$ the variation in price must be negative for the same reason. This effect can be translated into the timely development of the market price or better traded price. The (non-conserved) value for the buyer does not explicitly depend on time. (Else it would not be possible to choose the starting time as for example t = 0) So, we have $\mathrm{dvalue}/\mathrm{dt} = \partial\mathrm{value}/\partial\mathrm{p_m} \cdot \mathrm{dp_m}/\mathrm{dt}$. In other words we have:

$$\frac{dp_m}{dt} = \frac{1}{\frac{\partial value}{\partial p_m}} \cdot \frac{dvalue}{dt} \qquad [38]$$

If $\partial\mathrm{value}/\partial\mathrm{p_m} > 1$ and $\delta p > 0$, than the market price will increase slower (in time) than the value for the buyer. So, the buyer has even more reason to buy and will do so. The market price will go to infinity. If $\partial\mathrm{value}/\partial\mathrm{p_m} < 1$ and $\delta p < 0$, than the market price will decrease faster than the value for the buyer. So, it is allowed for the price to become smaller and smaller. Mathematically spoken it will reach minus infinity.

So, the price has an accelerated movement either to plus infinity or minus infinity. These are the only points the market tends to go by itself. Of course, neither a price of minus infinity nor of plus infinity can happen in reality. As argued in figure 31 already price will stay between p_f and p_u. This reasonable regime of prices may vary for every buyer. If either limit is reached by a “critical mass” of investors, the market will switch from rising to falling without any notice. In other words, it will behave chaotically. This is by no means different from the funny cartoon below:

Figure 33: Funny interpretation of statement 3 (source: The Economist)

In some sense I have given a mathematical proof of the crazy behavior of financial markets as it is "known" by most people. Or at least by so many people that there are already jokes about it (see figure 33).

To conclude this section I will formulate the following important statement:

Statement 3: Inevitability of chaos in financial markets

- → **In markets not governed by conserved quantities where the buyers' (non-conserved) value of the goods depends on (future) price, there are two stable points for the price only: minus and plus infinity. The price will run in one direction and change it without any other change. It therefore behaves chaotically.**

So there are two main ingredients responsible for chaos. One is the condition of non-conserved values and the other is a dependence on price of this (non-conserved) value. A market with strong price dependence, in a sense that higher prices are more attractive, is a fact in most markets of luxury goods. Whether the value of these luxury goods is conserved or not, depends on the particular case. For posh sports cars it is important to be expensive. But the cost of producing it is also very high. Therefore, value is conserved here and no chaos will happen. As an example consider tulips within the great tulip mania (Goldgar (2008)) as luxury goods. Their value was not conserved and the market was chaotic.

But even in the financial markets there may be segments were chaos plays no role. If somebody is buying stocks in order to take over (part) of a company or is investing in order to use the dividends as a retirement payments, all valuation of the stocks consider a conserved value. It is either the cash flow the merged companies will deliver or the expected future cash flow of the company itself which guaranties the dividends. Both kinds of investments are long term and even chaotic jumps in market prices of the stocks are completely irrelevant. Here one may speak of real investment as defined by Bogle (2012) or the dark grey area in figure 28.

In contrast, almost all of the total financial market consists of rapid buying and selling. There statement 3 is perfectly valid and chaos is always present. No predictions and no profits can be made (in the long run). It is identical to gambling.

IV. Appendix

This appendix consists of detailed argumentations and especially mathematical derivations which are important in their own right. They always refer to a certain content of the main text. However, they would make the corresponding chapters too clumsy.

1. Exact solution of warehouse location

The general question asked and partly answered here is where to place a warehouse in order to minimize the sum of transport and warehouse costs. In figure 34 the so called Steiner Weber

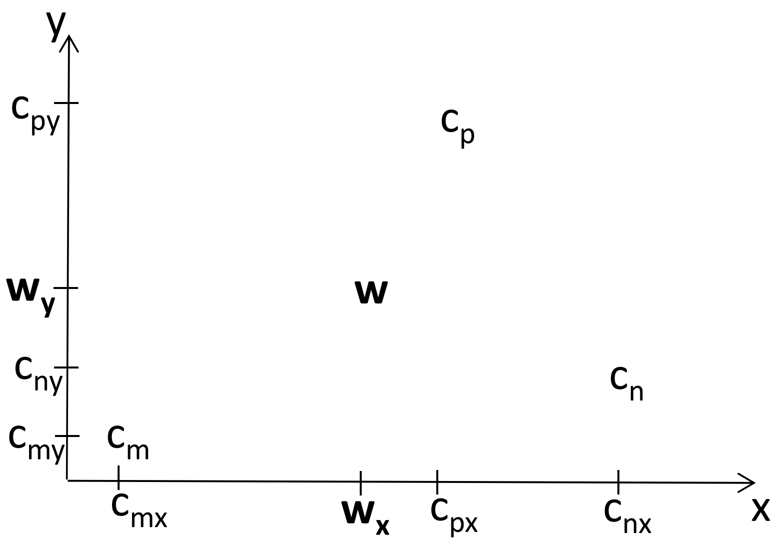

Figure 34: Generalized Steiner Weber problem

problem or better approach in a generalized form is displayed. There are n customers (just three of them c_n, c_m, and c_p are displayed in figure 34). They have the coordinates c_{ix} and c_{iy}. Each customer needs an amount of goods c_i (for example 100 kg per day of…). There is one warehouse w with coordinates w_x and w_y. In this model, one assumes that one drives on straight lines from the warehouse to the customers, but never between customers. (The latter method would be a travelling salesman problem.) If one assumes a specific transport cost t, so that the cost of transport to customer c_i is given by t times c_i times distance between c_i and w, then the total cost of transport is given by

Appendix

$$cost = t \cdot \sum_{i=1}^{n} c_i \cdot \sqrt{(w_x - c_{ix})^2 + (w_y - c_{iy})^2} \qquad [\ 39\ ]$$

To minimize the cost one may differentiate [39] with respect to w_x and w_y, respectively, and setting both equations to zero. So the cost minimum is received at a warehouse position (w_x, w_y), where the coordinates are received from solving the following set of equations:

$$\sum_{i=1}^{n} \frac{c_i \cdot (w_x - c_{ix})}{\sqrt{(w_x - c_{ix})^2 + (w_y - c_{iy})^2}} = 0 \ , \ \sum_{i=1}^{n} \frac{c_i \cdot (w_y - c_{iy})}{\sqrt{(w_x - c_{ix})^2 + (w_y - c_{iy})^2}} = 0 \qquad [\ 40\ ]$$

Solving [40] is straight forward. It leads to a polynomial of much higher power then n. Needless to say there is no general analytic solution. A numerical solution is simple, especially if modern computers and software are used. (For a numerical solution one should use [39] and perform a numerical minimization directly. Solving [40]numerically leads to many solutions, and one has to choose the correct one by inserting it into [39].) Special software for solving the problem numerically is available even for free download. For practical considerations one may include a route planner in order to calculate the cost for driving on real roads rather than straight lines. But this is not the goal of this thesis. For the special case of just two customers the solution is trivial. It may be found in Grabinski (2007) or Grabinski (2008). The warehouse is located at the biggest customer. But even for three customer (or more) there is no general analytic solution.

If all customers consume the same, the situation is much simpler. Mathematically spoken, all c_i's are identical and the c_i in [39] can be taken in front of the summation. In doing so [40] transforms to:

$$\sum_{i=1}^{n} \frac{w_x - c_{ix}}{\sqrt{(w_x - c_{ix})^2 + (w_y - c_{iy})^2}} = 0 \ , \ \sum_{i=1}^{n} \frac{w_y - c_{iy}}{\sqrt{(w_x - c_{ix})^2 + (w_y - c_{iy})^2}} = 0 \qquad [\ 41\ ]$$

Though [41] looks almost identical to [40] it is much simpler to solve. Again it is a polynomial of high order. However, due to its higher symmetry the polynomial does not have all powers. It may have even powers only. So the order can be reduced. (Like a biquadratic equation is of fourth order but soluble like an ordinary quadratic equation.) Please note that n customers define an n-corner surface. *However, the center of mass of it is not the optimal warehouse position* though it is close to it in many cases. Despite these nice simplifications analytical solutions for [41] are possible for $n < 4$ only. The only non-trivial solution is possible for three customers. This problem is often called the Fermat problem because Fermat found a geometric solution for it in around 1640 already. On the left hand side of figure 35 the three customers and their coordinates are displayed. The question is to find a warehouse position so that the sum of the distances $\overline{wc_1} + \overline{wc_2} + \overline{wc_3}$ becomes minimal. The right hand side of figure 35 denotes a triangle with the so called Fermat point F. The point is defined in a way that the three angles $\delta_{12} = \delta_{13} = \delta_{23} = 120°$. Fermat proved that the Fermat point is identical to the optimal warehouse location. Please note that it is only true if all angles of the triangle are smaller than 120°. If one angle becomes 120° or more, the Fermat point will be at the line opposite to angle.

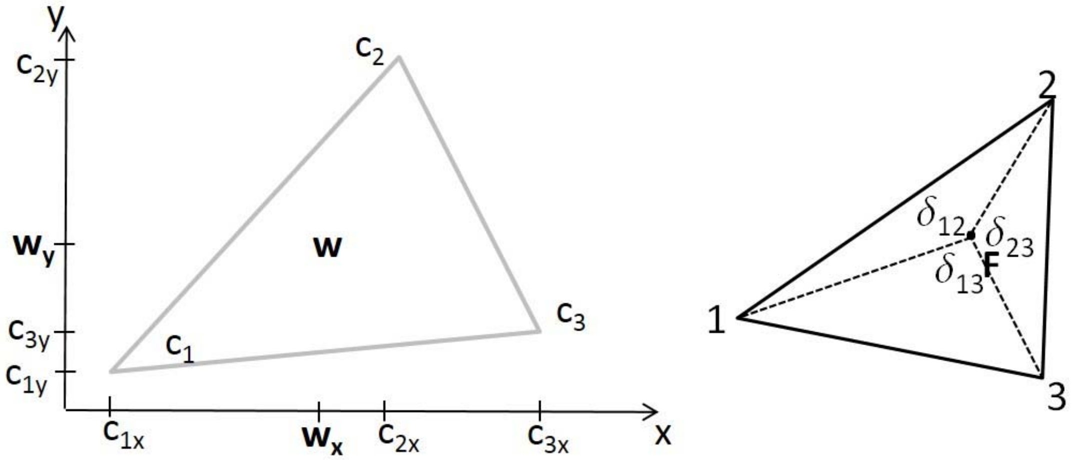

Figure 35: Fermat problem and its geometric solution

The geometric solution is quite simple. However, an analytic solution is pretty hard to find, and up to the best of my knowledge nobody ever did. Formulating the problem is pretty simple. Just taking [41] and allowing three customers only leads to:

$$\frac{w_x - c_{1x}}{\sqrt{(w_x - c_{1x})^2 + \left(w_y - c_{1y}\right)^2}} + \frac{w_x - c_{2x}}{\sqrt{(w_x - c_{2x})^2 + \left(w_y - c_{2y}\right)^2}} + \frac{w_x - c_{3x}}{\sqrt{(w_x - c_{3x})^2 + \left(w_y - c_{3y}\right)^2}} = 0$$

& [42]

$$\frac{w_y - c_{1y}}{\sqrt{(w_x - c_{1x})^2 + (w_y - c_{1y})^2}} + \frac{w_y - c_{2y}}{\sqrt{(w_x - c_{2x})^2 + (w_y - c_{2y})^2}} + \frac{w_y - c_{3y}}{\sqrt{(w_x - c_{3x})^2 + (w_y - c_{3y})^2}} = 0$$

There are several subsequent quadrations necessary in order to get rid of the roots. Then one should simplify and make proper substitutions in order to find a forth order polynomial. As one sees below, one can show that it is possible. However, doing so is incredible puzzling. Even if one uses smart software like Mathematica, a powerful workstation and a couple of thousands of hours of CPU time, no solution becomes feasible. Even if it would be possible, one would end up with four solutions. And it is extremely difficult to show which solution is the true one.

Therefore, another approach has been used successfully here. In order to get rid of variables one may transform the problem of figure 35 into the one of figure 36.Now customer one lies in the

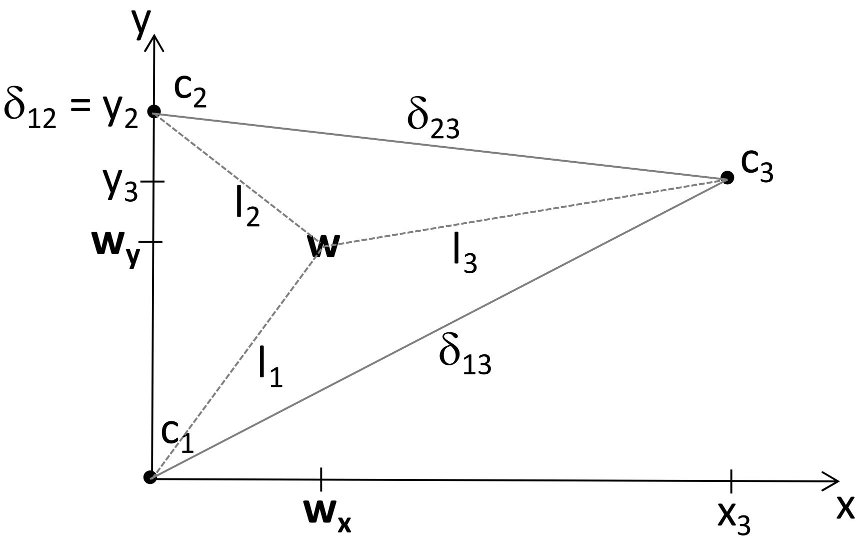

Figure 36: Fermat's Problem with smart coordinates

origin having the coordinates (0, 0). Customer two has the coordinates (0, y_2), and customer three has the coordinates (x_3, y_3). Of course, the problem in figure 36 is as general as the original Fermat problem (cf. figure 35). Mathematically the coordinates from figure 35 can be transformed by applying the transformation of [43] into the coordinates of figure 36.

$$+(-c_{1x}, -c_{1y}) \quad and \quad \begin{pmatrix} \frac{c_{2y}-c_{1y}}{y_2} & \frac{c_{1x}-c_{2x}}{y_2} \\ \frac{c_{2x}-c_{1x}}{y_2} & \frac{(c_{1x}-c_{2x})^2-y_2^2}{y_2(c_{1y}-c_{2y})} \end{pmatrix} \cdot \qquad [\ 43\ ]$$

The reverse transformation is displayed in [44]. With it, all results can be retransformed into the old coordinates:

$$\begin{pmatrix} \frac{(c_{1x}-c_{2x})^2-y_2^2}{(c_{1y}-c_{2y})y_2} & \frac{c_{2x}-c_{1x}}{y_2} \\ \frac{c_{1x}-c_{2x}}{y_2} & \frac{c_{2y}-c_{1y}}{y_2} \end{pmatrix} \cdot \quad and \quad +(c_{1x}, c_{1y}) \qquad [\ 44\ ]$$

Now the Steiner Weber Problem as stated in figure 36 can be solved. To the best of my knowledge such general solution has never been performed. Therefore, I will show the result and its derivation here. Applying Pythagoras' theorem to figure 36 yields:

$$w_x = \sqrt{{l_1}^2 - \left(\frac{{y_2}^2 + {l_1}^2 - {l_2}^2}{2 \cdot y_2}\right)^2} \qquad and \qquad w_y = \frac{{y_2}^2 + {l_1}^2 - {l_2}^2}{2 \cdot y_2} \qquad [\ 45\ ]$$

This is, of course, not the solution. It is still necessary to find expressions for l_1 and l_2 as functions of y_2, x_3, and y_3. From Fermat's geometric solution of the Steiner Weber problem for three customers one knows, that all angles around **w** of figure 36 are 120°. Using the cosines theorem for y_2, l_1, and l_2 and cos 120° = -½ one gets:

$${y_2}^2 = {l_1}^2 + {l_2}^2 + l_1 \cdot l_2 \qquad [\ 46\ ]$$

Using the cosines theorem in the same manner as in [46] for δ_{13} and δ_{23} yields:

$${\delta_{13}}^2 = {l_1}^2 + {l_3}^2 + l_1 \cdot l_3 \qquad and \qquad {\delta_{23}}^2 = {l_2}^2 + {l_3}^2 + l_2 \cdot l_3 \qquad [\ 47\ ]$$

Subtracting the second equation in [47] from the first yields:

$$l_3 = \frac{{\delta_{13}}^2 - {\delta_{23}}^2}{l_1 - l_2} - (l_1 + l_2) \qquad [\ 48\ ]$$

Using Pythagoras' theorem for δ_{13} and δ_{23} and building the difference of δ_{13} and δ_{23} yields:

$${\delta_{13}}^2 - {\delta_{23}}^2 = 2 \cdot y_2 \cdot y_3 - {y_2}^2 \qquad [\ 49\ ]$$

Inserting [49] into [48] yields:

$$l_3 = \frac{2 \cdot y_2 \cdot y_3 - {y_2}^2}{l_1 - l_2} - (l_1 + l_2) \qquad [\ 50\ ]$$

Pythagoras' theorem applied to δ_{13} gives ${\delta_{13}}^2 = {x_3}^2 + {y_3}^2$. For δ_{13} one may insert the first equation of [47] and for l_3 the expression of [50]. After some rearrangements one finds:

$$\begin{aligned}({x_3}^2 + {y_3}^2) \cdot (l_1 - l_2)^2 = {} & {l_1}^2 \cdot (l_1 - l_2)^2 + \left(2 \cdot y_2 \cdot y_3 - {y_2}^2 - (l_1 + l_2) \cdot (l_1 - l_2)\right)^2 \\ & + l_1 \cdot \left((2 \cdot y_2 \cdot y_3 - {y_2}^2) \cdot (l_1 - l_2) - (l_1 + l_2) \cdot (l_1 - l_2)^2\right)\end{aligned} \qquad [\ 51\ ]$$

With [46] and [51] there are two equations for the two variables l_1 and l_2. Inserting the solution for l_1 and l_2 into [45] gives the optimal warehouse position as long as all angles in the triangle of customers are smaller than 120°. Else the optimal warehouse position is on the side opposite to the angle ≥ 120°. (Mathematically it may take any position on this line. In real life situations one will choose the coordinates of one of the customers at the end of this line)

Please note that solving [46] and [51] is tedious but straightforward. Here the smart software Mathematica has been used. As it turns out, [46] and [51] are biquadratic in l_1 and l_2 leading to (quite lengthy) expressions for l_{11}, l_{12}, l_{13}, l_{14}, l_{21}, l_{22}, l_{23}, and l_{24} with the relations $l_{11} = -l_{12}$, , $l_{13} = -l_{14}$, $l_{21} = -l_{22}$, and $l_{23} = -l_{24}$. Negative l's are of course nonsense. Furthermore, w_x and w_y of [45] are quadratic in the l's. So, the sign does not matter or, in other words, $w_{x1} = w_{x2}$ and $w_{y1} = w_{y2}$ and $w_{x3} = w_{x4}$ and $w_{y3} = w_{y4}$. The explicit results are displayed in [53] on page 73. Having two solutions for the warehouse position refers to the fact that cos 120° = cos 240° = -½. I used cos 120° = - ½ in order to apply Fermat's geometric solution. The second solution for the warehouse position refers to a warehouse position outside the customer triangle with two angles equal to 120° and one angle equal to 240°. Unfortunately, depending on the customers' coordinates either one may apply. Deciding on one position is extremely difficult by using pure mathematics. Actually it is the reason why the software Mathematica uses thousands of hours of CPU time without finding the solution in one step (without using Fermat's geometric solution). In order to find the correct solution one may use the fact that the optimal warehouse position must be a continuous function of the customers' coordinates. (To prove it, consider Fermat's geometric solution. It is a unique solution as long as all angles are smaller than 120°. Therefore, the warehouse position can't "jump" by any variation of any parameters.) Using this argument a jump from one solution to another may happen only if the two solutions are equal at that point:

$$w_{x1} = w_{x3} \ and \ \ w_{y1} = w_{y3} \quad implies \quad x_3 = 0 \ \ or \ \ y_3 = {}^{y_2}/_{2} \qquad [\ 52\ ]$$

Though [52] in connection with [53] is very complicated, its solution is very simple. Please note that I am considering the right hand side of the coordinate system (x > 0) only. (The left hand side is the mirror image. To include it correctly a ± sign must be included in front of the root in [45].) Therefore, the solution for the optimal warehouse position changes from index 1 or 2 to 3 or 4 if y_3 gets bigger or smaller than $y_2/2$. In order to see it one may make a three dimensional plot of both functions $w_{y1} = w_{y2}$ and $w_{y3} = w_{y4}$. In figure 37 I have chosen y2 = 1. That is by no means a limitation, but it makes the three dimensional rather than four dimensional plot possible. (Formally, one could have introduced a factor of 1/y2 or y2 in front of the matrixes of [43] or [44], respectively.)

$$w_{x1} = w_{x2} = \frac{\sqrt{\frac{y_2^4(x_3^2 - 3y_3^2)^2(x_3^6y_2^2 + x_3^4y_2^2(3y_2^2 - 20y_3y_2 + 19y_3^2) + x_3^2(2\sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 + y_3^2 - 2y_2y_3)^2} + 3y_3(-2y_2^3 + 16y_3y_2^2 - 24y_3^2y_2 + 9y_3^3)y_2^2) - 3(y_2 - y_3)y_3(2\sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 + y_3^2 - 2y_2y_3)^2} + 3y_2^2(2y_2^2 - 3y_3y_2 + y_3^2)y_3^2))}{(x_3^2 + y_3(y_3 - 2y_2))(y_2^2(x_3^2(y_2^2 - 7y_3y_2 + 4y_3^2) + x_3^4 + 3y_3^2(y_2^2 - y_3y_2 + y_3^2)) + \sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 + y_3^2 - 2y_2y_3)^2})^2}}}{2\sqrt{3}}$$

$$w_{y1} = w_{y2} = \frac{y_2(x_3^2 - 3y_3^2)(3y_2^2(x_3^2 + y_3(y_3 - 2y_2))(x_3^2 - y_3(y_2 + y_3)) + \sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 + y_3^2 - 2y_2y_3)^2})}{6(x_3^2 + y_3(y_3 - 2y_2))(y_2^2(x_3^2(y_2^2 - 7y_3y_2 + 4y_3^2) + x_3^4 + 3y_3^2(y_2^2 - y_3y_2 + y_3^2)) + \sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 + y_3^2 - 2y_2y_3)^2})}$$

$$w_{x3} = w_{x4} = \frac{\sqrt{-\frac{y_2^4(x_3^2 - 3y_3^2)^2(-x_3^6y_2^2 + x_3^4y_2^2(-3y_2^2 + 20y_2y_3 - 19y_3^2) + 3y_3(y_2 - y_3)(3y_2^2y_3^2(2y_2^2 - 3y_2y_3 + y_3^2) - 2\sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 - 2y_2y_3 + y_3^2)^2}) + x_3^2(2\sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 - 2y_2y_3 + y_3^2)^2} + 3y_2^2y_3(2y_2^3 - 16y_2^2y_3 + 24y_2y_3^2 - 9y_3^3)))}{(x_3^2 + y_3(y_3 - 2y_2))(\sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 - 2y_2y_3 + y_3^2)^2} - y_2^2(x_3^4 + x_3^2(y_2^2 - 7y_2y_3 + 4y_3^2) + 3y_3^2(y_2^2 - y_2y_3 + y_3^2)))^2}}}{2\sqrt{3}}$$

$$w_{y3} = w_{y4} = -\frac{y_2(x_3^2 - 3y_3^2)(\sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 + y_3^2 - 2y_2y_3)^2} - 3y_2^2(x_3^2 + (y_3 - 2y_2)y_3)(x_3^2 - y_3(y_2 + y_3)))}{6(x_3^2 + (y_3 - 2y_2)y_3)(y_2^2(x_3^2(y_2^2 - 7y_3y_2 + 4y_3^2) + x_3^4 + 3y_3^2(y_2^2 - y_3y_2 + y_3^2)) - \sqrt{3}\sqrt{x_3^2y_2^4(y_2 - 2y_3)^2(x_3^2 + y_3^2 - 2y_2y_3)^2})}$$

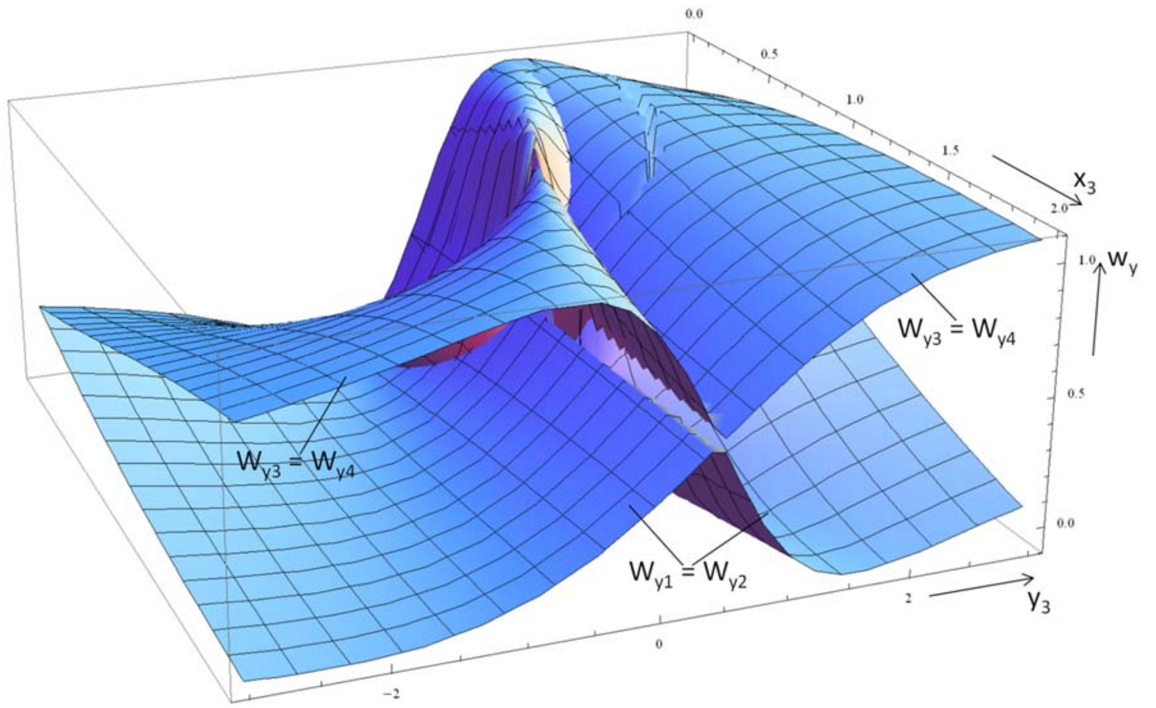

Figure 37: Warehouse coordinate w_y as function of customer 3

For bigger y_3 the warehouse coordinate w_y must also increase and for smaller values of y_3 it must decrease. In other words, w_y is a monotonous function of y_3. From figure 37 one sees that for $y_3 > ½$ is $\overrightarrow{w_3} = \overrightarrow{w_4}$ the optimal warehouse position and for $y_3 < ½$ it is $\overrightarrow{w_1} = \overrightarrow{w_2}$. In general, one can say *that the optimal warehouse position, as stated in [53], is given by the index 1 or 2 for a y-position of customer 3 smaller than half of the y-position of customer 2. For $y_3 > y_2/2$ the indices 3 or 4 are good*.

2. Approximation of warehouse location used here

In the last chapter I have shown the solution for an optimal warehouse position with three equal customers. For the considerations of chapter 1 it is necessary to consider customers with different consumption rates. Though such generalization is possible with the solution of the previous chapter, it is extremely tedious. In order to have a result that is easier to understand, the special

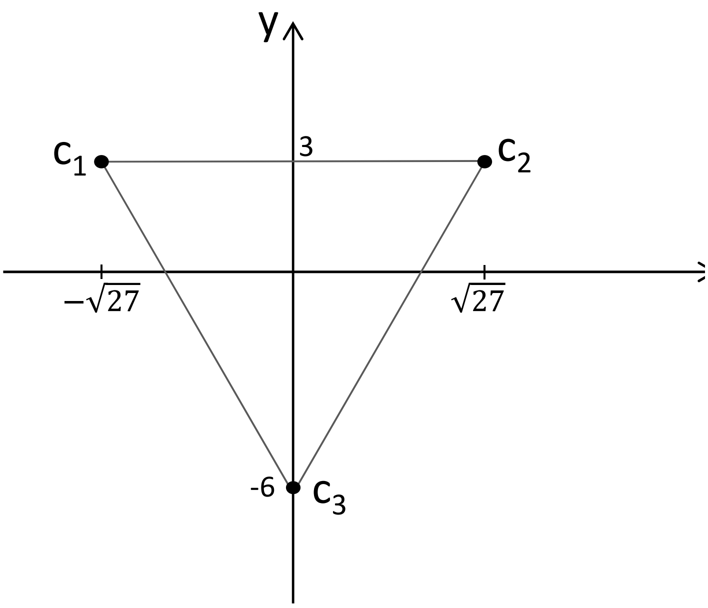

Figure 38: Approximation of three customers in a regular triangle

case of three customers in a regular triangle (cf. figure 38) is considered here. Such approximation does not cause a principle difference for the effects of chaos discussed in this thesis. Furthermore, a special length scale has been introduced in figure 38. This is of course no approximation; it is a result of a transformation. In doing so one gets rid of superfluous variables.

For an arrangement like the one of figure 38 the total cost of transport is given by (cf. [39])

$$cost = t \cdot \left(c_1 \cdot \sqrt{\left(w_x + \sqrt{27}\right)^2 + \left(w_y - 3\right)^2} + c_2 \cdot \sqrt{\left(w_x - \sqrt{27}\right)^2 + \left(w_y - 3\right)^2} \right) + c_3 \cdot \sqrt{{w_x}^2 + \left(w_y + 6\right)^2} \qquad [\ 54\ ]$$

In order to find the optimal warehouse position one has to differentiate [54] with respect to w_x and w_y and setting both functions to zero. Doing so leads to (cf. [40]):

Appendix

$$c_1 \cdot \frac{w_{x\,opt} + \sqrt{27}}{\sqrt{\left(w_{x\,opt} + \sqrt{27}\right)^2 + \left(w_{y\,opt} - 3\right)^2}} + c_2 \cdot \frac{w_{x\,opt} - \sqrt{27}}{\sqrt{\left(w_{x\,opt} - \sqrt{27}\right)^2 + \left(w_{y\,opt} - 3\right)^2}} + c_3 \cdot \frac{w_{x\,opt}}{\sqrt{{w_{x\,opt}}^2 + \left(w_{y\,opt} + 6\right)^2}} = 0$$

$$c_1 \cdot \frac{w_{y\,opt} - 3}{\sqrt{\left(w_{x\,opt} + \sqrt{27}\right)^2 + \left(w_{y\,opt} - 3\right)^2}} + c_2 \cdot \frac{w_{y\,opt} - 3}{\sqrt{\left(w_{x\,opt} - \sqrt{27}\right)^2 + \left(w_{y\,opt} - 3\right)^2}} + c_3 \cdot \frac{w_{y\,opt} + 6}{\sqrt{{w_{x\,opt}}^2 + \left(w_{y\,opt} + 6\right)^2}} = 0 \quad [\ 55\ ]$$

Though [55] is much simpler than [40], it has no general solution. For the special case $c_1 = c_2 = c_3 \equiv c$ the solution is $w_{x\,opt} = w_{y\,opt} = 0$. The optimal warehouse position lies at the origin. Of course, the coordinates of figure 38 have been chosen deliberate in this way. Please note that an optimal warehouse position is (for equal customers) always in the center of a regularly shaped figure. Having the solutions for equal customers one may now set

$$c_1 = c \cdot (1 - \varepsilon) \quad and \quad c_2 = c \cdot (1 + \delta) \quad and \quad c_3 = c \qquad (\varepsilon, \delta > 0) \quad [\ 56\ ]$$

In the same manner the optimal warehouse position can be understood as a development around the origin:

$$w_{x\,opt} = 0 + x \quad and \quad w_{y\,opt} = 0 + y \quad [\ 57\ ]$$

Of course, the zeros in front of the plus signs in [57] can be skipped. Now one can insert [57] and [56] into [55] leading to:

$$(1-\varepsilon) \cdot \frac{x + \sqrt{27}}{\sqrt{\left(x + \sqrt{27}\right)^2 + (y-3)^2}} + (1+\delta) \cdot \frac{x - \sqrt{27}}{\sqrt{\left(x - \sqrt{27}\right)^2 + (y-3)^2}} + \frac{x}{\sqrt{x^2 + (y+6)^2}} = 0$$

$$(1-\varepsilon) \cdot \frac{y - 3}{\sqrt{\left(x + \sqrt{27}\right)^2 + (y-3)^2}} + (1+\delta) \cdot \frac{y - 3}{\sqrt{\left(x - \sqrt{27}\right)^2 + (y-3)^2}} + \frac{y + 6}{\sqrt{x^2 + (y+6)^2}} = 0 \quad [\ 58\ ]$$

[58] is still as general as [55]. Therefore it has no general solution. As already indicated one may consider this approach as a development around equal customers. Then a procedure equal to perturbation theory in physics may start. One can expand all terms in [58] in x and y (and ε and δ) by using Taylor series.

$$\frac{1}{\sqrt{\left(x \pm \sqrt{27}\right)^2 + (y-3)^2}} = \frac{1}{6} \cdot \left(1 + \frac{y}{12} \mp \frac{x}{4\sqrt{3}} + O(x,y)^2\right)$$

$$\frac{1}{\sqrt{x^2 + (y+6)^2}} = \frac{1}{6} \cdot \left(1 + \frac{y}{6} + O(x,y)^2\right) \quad [\ 59\ ]$$

For the roots of [60] one gets in first order of x and y the series of [59]. They can be inserted into [58]. In lowest order it yields:

$$w_{x\,opt} = x = 2\sqrt{3}\cdot(\delta+\varepsilon) + O(\delta,\varepsilon)^3 \quad and \quad w_{y\,opt} = y = 2\cdot(\delta-\varepsilon) + O(\delta,\varepsilon)^3 \qquad [\ 60\ ]$$

Please note that [60] is good up to second order in δ and ε though the original expansion was up to first order only. Due to the anti-symmetry for $\delta \to -\delta$ and $\varepsilon \to -\varepsilon$ there must not be any even powers in [60] (they must be zero). By using [56] the optimal warehouse position of [60] can also be expressed with the original c_1, c_2, and c_3. By inserting the optimal warehouse position of [60] into [54] (total cost) and using the Taylor series

$$\sqrt{\left(x \pm \sqrt{27}\right)^2 + (y-3)^2} = 6\cdot\left(1 - \frac{y}{12} \pm \frac{x}{4\sqrt{3}} + O(x,y)^2\right)$$
$$\sqrt{x^2 + (y+6)^2} = 6\cdot\left(1 - \frac{y}{6} + O(x,y)^2\right) \qquad [\ 61\ ]$$

for the roots (like in [61]) one gets for the total cost:

$$cost = t\cdot c\cdot\left(1 + \frac{\delta}{3} - \frac{\varepsilon}{3} - \frac{2\delta^2}{9} - \frac{2\delta\varepsilon}{9} - \frac{2\varepsilon^2}{9} + O(\delta,\varepsilon)^3\right) \qquad [\ 62\ ]$$

By using [56] the total cost can be expressed by the original c_1, c_2, and c_3. Adding the cost for one warehouse f_w to it leads to the total cost for one warehouse c_{one}:

$$c_{one} = f_w + t\cdot 18c_3\cdot\left(1 + \frac{1}{3}\cdot\frac{c_1 - c_3 + c_2 - c_3}{c_3} - \frac{2}{9}\cdot\left(\frac{c_2 - c_1}{c_3}\right)^2 + \frac{2}{9}\cdot\frac{c_3 - c_1}{c_3}\frac{c_2 - c_3}{c_3} + O(\Delta c)^3\right) \qquad [\ 63\ ]$$

It is an expansion in differences of c_i with $\Delta c \equiv (c_i - c_j)/c_k$ with $i \neq j \neq k$.

Having two warehouses instead of one it is clear to put the one warehouse at the biggest customer (here C_2) and the second warehouse will support the other two customers. The optimal warehouse position for two customers is quite simple and can be found in for example Grabinski (2007) or Grabinski (2008). The second warehouse has to be placed at the second biggest customer (here C_3). This warehouse transports goods to C_1 over a distance $\sqrt{108}$ (cf. figure 38). It leads to a total cost of:

$$c_{two} = 2\cdot f_w + 6\sqrt{3}\cdot c_1\cdot t \qquad [\ 64\ ]$$

Chaos is present in a regime where $c_{one} = c_{two}$, because an arbitrarily small change in any c_i will result in a change from one warehouse to two warehouses. By using [63] and [64] $c_{one} = c_{two}$ leads to a quadratic equation in the c_i. Because all c_i are positive one finds for example:

$$c_3 = \frac{f_w}{4t} + \frac{3}{2}\sqrt{3}c_1 - \frac{5}{2}(c_1 + c_2) + \frac{1}{4}\sqrt{(-6\sqrt{3}c_1 + 10c_1 + 10c_2 - f_w)^2 - 8(-4c_1^2 + 4c_2c_1 - 4c_2^2)} + O(\Delta c)^3 \qquad [\ 65\ ]$$

Please note that the equation is only good up to second order in Δc. Consequently the root in [65] should be expanded in Δc-terms. However, this does not really lead to a simplification and is therefore omitted. Furthermore we have the restriction (due to [56]) $c_2 > c_3 > c_1$.

At least, when the consumption rates c_i become very big, it is cheaper to have three warehouses. Obviously they are placed at each customer. It leads to total costs of:

$$c_{three} = 3 \cdot f_w \qquad [\ 66\ ]$$

Equating c_{two} from [64] with c_{three} from [66] leads to a (chaotic) transition from two warehouses to three at:

$$c_1 = \frac{\sqrt{3}}{18} \cdot \frac{f_w}{t} \qquad [\ 67\ ]$$

Again there is the restriction $c_2 > c_3 > c_1$. Inserting [67] into [64] leads to a "triple point" where $c_{one} = c_{two} = c_{three}$. Please note however, that [65] and [67] just state regimes where the total cost for optimally placed warehouses is equal, though the number of warehouses changes. Furthermore, for small consumption rates one warehouse is best and for very huge consumption rates three warehouses are optimal. So it appears to be logical that in between two warehouses are optimal. But that is not necessary the case. Especially here there is a very small regime where two warehouses are optimal. Mostly either one or three warehouses are optimal. To find out what is true one has to compare the corresponding total costs of [63], [64], and [66].

3. Field theoretical approach to warehouse location

In the last section of the appendix I have considered three customers. A generalization to four customers is quite simple. Going to five and more customers is also possible but pretty tedious. In any case, considering a few customers is perfect to show the principle. However, it is hardly a real situation where the customers are plentiful. (It is a real situation for a warehouse bringing material to a couple of factory sites. However, in such arrangements the warehouse is almost always located at one of the factory sites for other reasons.)

In order to show a reasonable way how to manage many customers, another approach is suggested here. Instead of single customers I suggest to have a field of customers or a customer density c(x,y) defined so that the (differential) material current dj is given by:

$$dj = dx\ dy\ c(x,y) \quad or \quad j = \iint_S dx\ dy\ c(x,y) \qquad [\ 68\ ]$$

The total mass current j in an area S is given by the corresponding integration. Please note that this approach is not an approximation. It includes the situation having n distinct customers C_i with i=1,…,n at coordinates (x_i, y_i). It is easily seen if one introduces the so called δ-distribution (often referred to as δ-function) defined by:

$$\delta(a,b) = \delta(a) \cdot \delta(b) \quad and \quad \delta(x) = \begin{cases} \to \infty, & \lim\limits_{x \to 0} \\ 0, & x \neq 0 \end{cases} \quad and \quad \int_{-\infty}^{\infty} dx\ \delta(x) = 1 \qquad [\ 69\ ]$$

With the mathematical theorem that

$$\iint_S dx\ dy\ f(x,y) \cdot \delta(x-a, y-b) = f(a,b) \quad if\ \ (a,b)\ lies\ within\ S\,(else = 0) \qquad [\ 70\ ]$$

by setting

$$c(x,y) = \sum_{i=1}^{n} c_i \cdot \delta(x - x_i, y - y_i)) \qquad [\ 71\ ]$$

all integrals will turn into summation signs and one will be back to the situation of [39] or [40], etc.

In the next subsection I will discuss the most general approach. As it turns out, it will involve pretty messy mathematics especially if one considers more than one warehouse. But it is possible to see that chaos will not vanish by considering a field of customers. In order to see how it really works an unrealistic but mathematically simple approach is considered in subsection 3.3.2.

3.1 General approach (2 D)

A customer density distribution c(x,y) is considered here. In order to see the point I will give an example of it:

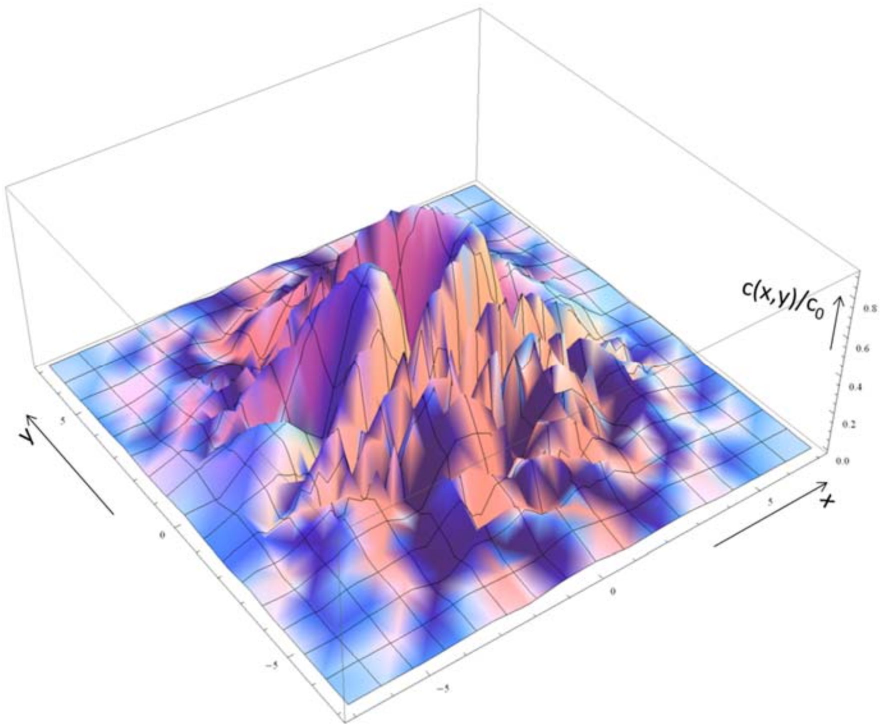

Figure 39: Example of customer density distribution

In figure 39 customer density has been plotted over the coordinates (x,y) which are in arbitrary units. So the density is highest around the center (0,0) and vanishes rapidly. (The toy example I have chosen here is $c(x,y) = c_0 \cdot |\sin(y^2 + x)| \cdot \exp\left(-1/15\,(x^2 + y^2)\right)$.) The total loads to be transported are given by:

$$\iint_{-\infty}^{\infty} dxdy\ c(x,y) \qquad (\approx 20.0014 \cdot c_0 \quad \textit{here in this example}) \qquad [\ 72\ ]$$

Please note that the customer density c(x,y) must decrease rapidly enough for convergence of the integrals. Alternatively one may choose a certain area to be delivered. Then the integrals transform into a two dimensional surface integral over a closed (finite) surface. Having a warehouse at (w_x, w_y) and a specific transport cost factor t (cf. [39] on page 68) the total cost of transportation is given by

$$cost = t \cdot \iint_{-\infty}^{\infty} dxdy\ c(x,y) \cdot \sqrt{(x-w_x)^2 + (y-w_y)^2} \qquad [\ 73\ ]$$

The cost of [73] must be minimal for optimal warehouse position. Depending on the customer density c(x,y) there are several possibilities to find this minimum. In general, there is a (complicated) numerical solution only. By taking the derivatives of [73] with respect to w_x and w_y and setting them to zero one finds:

$$\iint_{-\infty}^{\infty} dxdy\ c(x,y) \cdot \frac{x - w_{x,opt}}{\sqrt{(x-w_x)^2 + \left(y-w_y\right)^2}} = 0 \qquad and$$

$$\iint_{-\infty}^{\infty} dxdy\ c(x,y) \cdot \frac{y - w_{y,opt}}{\sqrt{(x-w_x)^2 + \left(y-w_y\right)^2}} = 0 \qquad [\ 74\ ]$$

In the toy example above the transport cost

$$t \cdot \iint_{-\infty}^{\infty} dxdy\ \mathrm{c_0} \cdot |\sin(\mathrm{y}^2 + \mathrm{x})| \cdot \exp(-1/15\,(\mathrm{x}^2 + \mathrm{y}^2)) \cdot \sqrt{(x-w_x)^2 + (y-w_y)^2} \qquad [\ 75\ ]$$

becomes minimal for a warehouse position at (roughly) w_{opt} = (0.0478, 0) as a numerical calculation shows. To get the result it is the simplest way to set $w_{y,opt}$ = 0 for symmetry reasons and then to calculate [75] numerically for various positions w_x. The total transport cost at that position is roughly $56.28658 \cdot c_0$ · t. The details of a graphical solution are shown in figure 40.

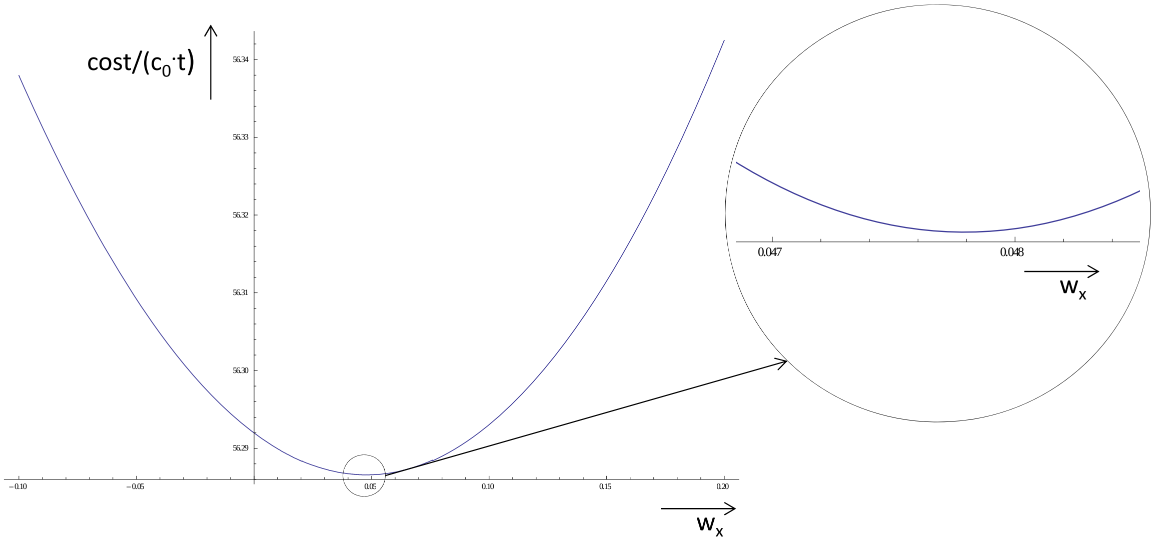

Figure 40: Total transport cost in 2D customer field over warehouse position w_x (w_y = 0)

Finding the optimal warehouse position for one warehouse is relatively simple. For two warehouses it is much more complicated. This is due to the difficulties to decide which warehouse should deliver which area for lowest transport cost.

Having two warehouses w_1 and w_2 one has to decide which warehouse should deliver to which customers for lowest cost. The situation is displayed in figure 41. There two surface areas S_1 and S_2 are displayed. The areas cover the delivery areas for the warehouses w_1 and w_2, respectively.

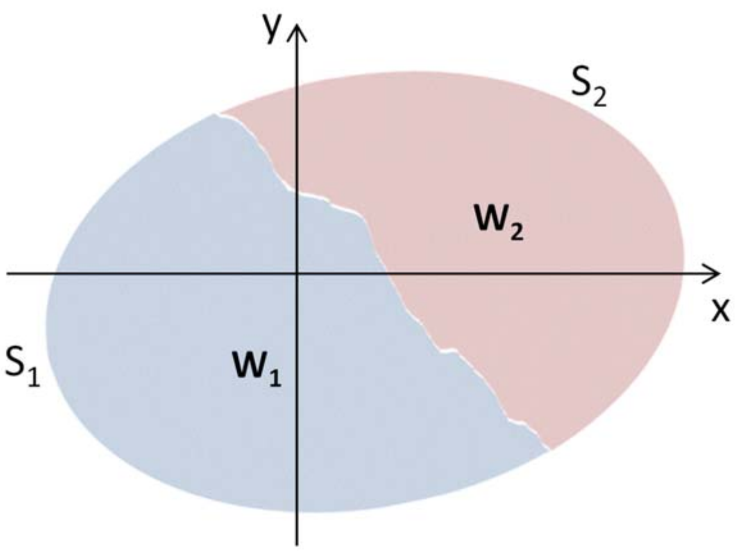

Figure 41: Delivery areas for two warehouses in a customer field c(x,y)

(The customer distribution field c(x,y) is not shown.) The areas are S_1 and S_2 are finite in figure 41. It is by no means a limitation as c(x,y) must vanish for x,y $\rightarrow \pm\infty$ anyway. The main question is to find the optimal line between the two warehouses displayed white in figure 41. The total cost of transport is given by:

$$cost = t\cdot\left(\iint_{S_1} dxdy\, c(x,y)\cdot\sqrt{(x-w_{1x})^2+(y-w_{1y})^2} + \iint_{S_2} dxdy\, c(x,y)\cdot\sqrt{(x-w_{2x})^2+(y-w_{2y})^2}\right) \quad [\,76\,]$$

In order to find the lowest cost, [76] must be optimized by the warehouse coordinates w_{1x}, w_{1y}, w_{2x}, w_{2y} *and* the surface areas S_1 and S_2. If the white line separating S_1 and S_2 in figure 41 is given by an (unknown) function y(x), the surface integrals of [76] can be written as:

$$cost = t\cdot\left(\int_{-\infty}^{\infty} dx\,\theta(y(x)-x)\int_{-\infty}^{y(x)} dy\ c(x,y)\cdot\sqrt{(x-w_{1x})^2+\left(y-w_{1y}\right)^2} + \int_{-\infty}^{\infty} dx\,\theta\big(x-y(x)\big)\int_{y(x)}^{\infty} dy\ c(x,y)\cdot\sqrt{(x-w_{2x})^2+(y-w_{2y})^2}\right) \quad [\,77\,]$$

Here $\theta(x)$ denotes the Heaviside function being 0 for x < 0 and 1 for x > 0. First one has to optimize with respect to y(x). Let $I_1(x,y)$ and $I_2(x,y)$ be defined by

$$\frac{\partial I_1}{\partial y} = c(x,y)\cdot\sqrt{(x-w_{1x})^2+\left(y-w_{1y}\right)^2}\ and\ \frac{\partial I_2}{\partial y} = c(x,y)\cdot\sqrt{(x-w_{2x})^2+(y-w_{2y})^2} \quad [\,78\,]$$

With this definition [77] turns into

$$cost = t\cdot\left(\int_{-\infty}^{\infty} dx\,\theta(y(x)-x)\cdot[I_1(x,y(x))-I_1(x,-\infty)] + \int_{-\infty}^{\infty} dx\,\theta\big(x-y(x)\big)\cdot[I_2(x,\infty)-I_2(x,y(x))]\right) \quad [\,79\,]$$

It is a very simple type of the Euler Lagrange functional. It is minimized if the derivative of the function under the integral with respect to y vanishes (It can be easily seen if one adds a small variation δ to y. Now one can expand the function under the integral into a Taylor series in powers of δ stopping at lowest order. Because cost must increase for any variation, be it positive or negative, the factor in front of δ must be zero.) Taking the derivative with respect to y under the Integral of [79], using [78], and the fact that c(x,y) vanishes at infinity one will find:

$$c\delta(y-x)\cdot[I_1(x,y)+I_2(x,y)-I_1(x,-\infty)-I_2(x,\infty)]$$
$$+\theta(y-x)\cdot c(x,y)\cdot\left(\sqrt{(x-w_{1x})^2+\left(y-w_{1y}\right)^2}-\sqrt{(x-w_{2x})^2+\left(y-w_{2y}\right)^2}\right)=0 \qquad [\ 80\ ]$$

δ denotes again the delta distribution (cf. [70]) with δ(x) = δ(-x), and θ is the Heaviside function with ∂_x θ (x) = δ(x) (for a simple proof of the latter one see [84]). [80] is (together with [78]) a highly non-linear differential equation for the function y(x). y(x) is the line separating the distribution areas S_1 and S_2 of w_1 and w_2 in figure 41. Even numerical this is a very tough problem for almost any given c(x,y). Opting for chaos it is even necessary to consider a small, for example, Gaussian fluctuation of c(x,y) in order to see whether the warehouse distribution will vary in accordance or behaves (almost) random. Doing so leads to coupled, stochastic, and non-linear differential equations. Arguably, it is still the greatest challenge in numerical mathematics.

After solving for y(x) one has to insert it into [77] and taking the derivatives with respect to w_{1x}, w_{1y}, w_{2x}, and w_{2y} and setting them to zero:

$$\int_{-\infty}^{\infty} dx\ \theta(y(x)-x)\int_{-\infty}^{y(x)} dy\ \ c(x,y)\cdot\frac{x-w_{1x\,opt}}{\sqrt{\left(x-w_{1x\,opt}\right)^2+\left(y-w_{1y\,opt}\right)^2}}=0$$
$$\int_{-\infty}^{\infty} dx\ \theta(y(x)-x)\int_{-\infty}^{y(x)} dy\ \ c(x,y)\cdot\frac{y-w_{1y\,opt}}{\sqrt{\left(x-w_{1x\,opt}\right)^2+\left(y-w_{1y\,opt}\right)^2}}=0$$
$$\int_{-\infty}^{\infty} dx\ \theta(y(x)-x)\int_{-\infty}^{y(x)} dy\ \ c(x,y)\cdot\frac{x-w_{2x\,opt}}{\sqrt{\left(x-w_{2x\,opt}\right)^2+\left(y-w_{2y\,opt}\right)^2}}=0 \qquad [\ 81\ ]$$
$$\int_{-\infty}^{\infty} dx\ \theta(y(x)-x)\int_{-\infty}^{y(x)} dy\ \ c(x,y)\cdot\frac{y-w_{2y\,opt}}{\sqrt{\left(x-w_{2x\,opt}\right)^2+\left(y-w_{2y\,opt}\right)^2}}=0$$

Please note that y(x) depends on w_{1x}, w_{1y}, w_{2x}, and w_{2y}. Therefore, [81] is much harder to solve than it appears already. Needless to say that I will not even try to solve it with the toy example from above.

Be it as it may, there will be a solution for the optimal warehouse positions and a corresponding total transport cost as given by [77]. If this cost plus the cost for two warehouses is smaller than the cost for one warehouse and its corresponding total transport cost (see [73]), then two warehouses are cheaper than one. This switch will appear within an arbitrarily small variation of c(x,y). In other words, one will have a chaotic behavior.

3.2 One dimensional approximation

In the previous subsection I have shown how to solve the field theoretical approach in principle. Especially for readers not too experienced in mathematics it might have been very confusing. Therefore, I will give the oversimplified example of a one dimensional customer distribution c(x). Though all customers are on one straight line now, they will be delivered by separate tours. This is in accordance to the star wise deliveries I have assumed in the two dimensional customer field. (Else the optimal warehouse position will be trivially at the maximum of c(x).)

An example for the customer density c(x) has been plotted in figure 42.

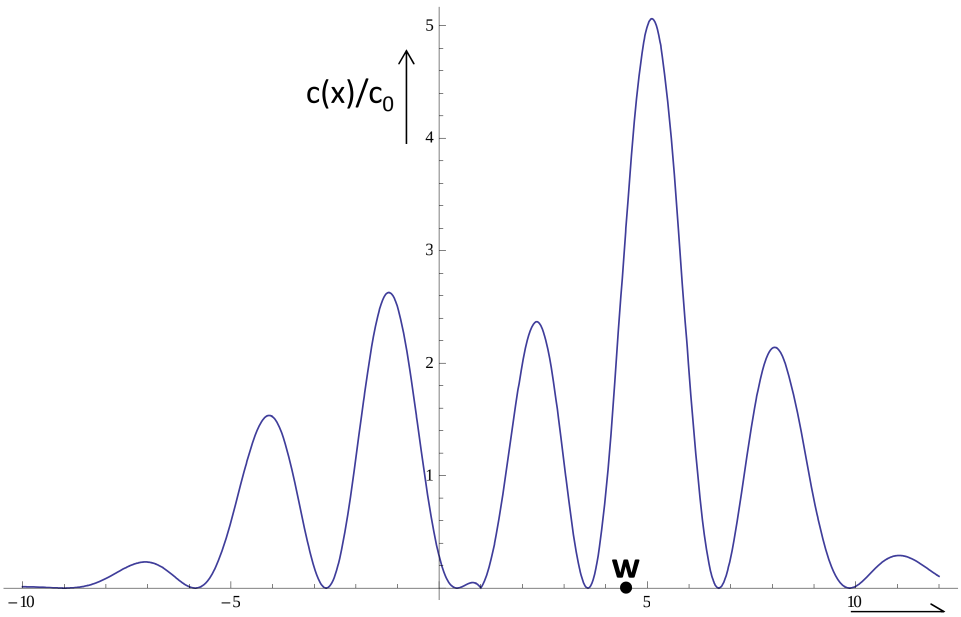

Figure 42: One dimensional customer density c(x)

The density varies over the coordinate x (in arbitrary units), and vanishes rapidly for x $\rightarrow \pm\infty$. The total amount to be transported is given by integration, in the case of figure 42 it is about 21.1538 c_0. If the warehouse is at x = w, the total cost of transport is given by:

$$t \cdot \iint_{-\infty}^{\infty} dxdy \; c_0 \cdot |\sin(y^2 + x)| \cdot \exp(-1/15\,(x^2 + y^2)) \cdot \sqrt{(x - w_x)^2 + (y - w_y)^2} \qquad [\,82\,]$$

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Depending on the customer density distribution c(x), it is a more or less smooth function of warehouse position w. In the case of the density of figure 42 it can be plotted as follows:

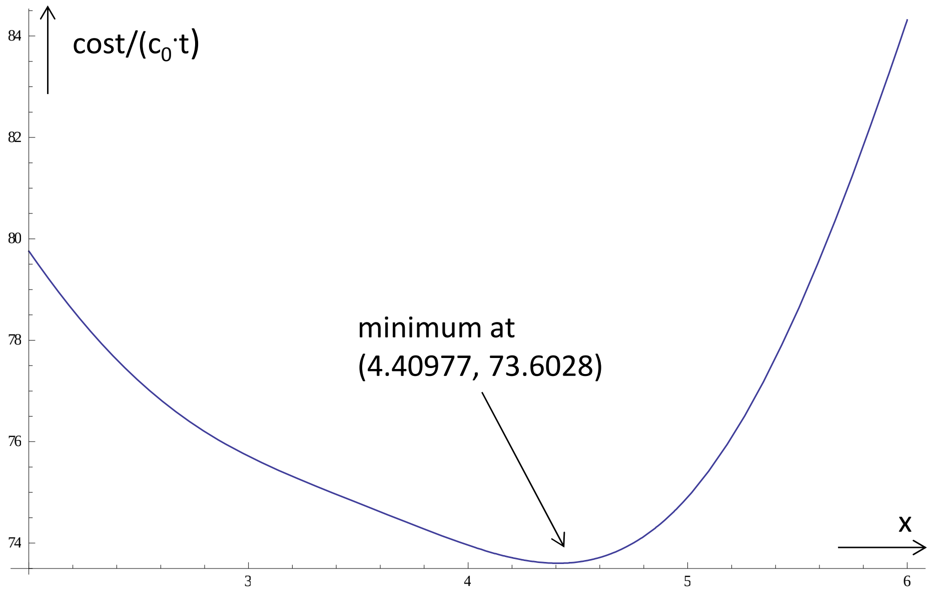

Figure 43: Total cost of transport in 1 D customer density plotted over warehouse position w

In order to calculate the optimal warehouse position w_{opt} one has to differentiate [82] with respect to w and setting it to zero. Because the differential of the absolute value of x – w is nothing but the signum function, the optimal warehouse position is determined by:

$$\int_{-\infty}^{w_{opt}} dx\, c(x) = \int_{w_{opt}}^{\infty} dx\, c(x) \qquad [\ 83\ ]$$

It is nothing but the "center" of the customer density distribution c(x). (Like a center of mass if c(x) was a mass distribution.) Please note that the limit to separate customers is possible but tricky here. Having, for example, two customers at positions c_1 and c_2 with consumption rates c_{01} and c_{02}, c(x) becomes equal to $c_{01}\cdot\delta(x-c_1) + c_{02}\cdot\delta(x-c_2)$. If $c_{01} = c_{02}$ [83] holds for every warehouse position $c_1 \leq w_{opt} \leq c_2$. Like already shown in Grabinski (2007, 2008), it may be on an arbitrary position between the two warehouses. However, for say $c_{01} < c_{02}$ it should be at the biggest customer ($w_{opt} = c_2$). In order to build this limit one has to write the delta distribution as a limit of another function, for example:

$$\delta(x) = \lim_{\varepsilon\to 0} \frac{\theta\left(x+\frac{\varepsilon}{2}\right) - \theta\left(x-\frac{\varepsilon}{2}\right)}{\varepsilon} \qquad [\ 84\ ]$$

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Inserting [84] into [83] one has to choose $w_{opt} = c_2 + \varepsilon/2 - \varepsilon \cdot (c_{02} - c_{01})/c_{02}$. The limit $\varepsilon \rightarrow 0$ then gives $w_{opt} = c_2$.

Having two warehouses at w_1 and w_2 (without any limitation $w_1 < w_2$) one has to decide what are the areas for delivery for each warehouse. Let the first warehouse deliver for x < a, and the second for x > a. The total cost of transport is then given by:

$$cost = t \cdot \left(\int_{-\infty}^{a} dx\, c(x) \cdot |x - w_1| + \int_{a}^{\infty} dx\, c(x) \cdot |x - w_2| \right) \qquad [\ 85\ ]$$

In order to find the optimal delivery regime one has to differentiate [85] with respect to "a" and setting the derivative to zero. The derivative of an integral with respect to an upper or lower limit gives plus or minus of the function under the integral at the value of the corresponding limit. Therefore one gets for "a":

$$|a - w_1| = |a - w_2| \quad or \quad a = \frac{w_1 + w_2}{2} \qquad [\ 86\ ]$$

In other words, each warehouse has to serve up to the middle between them. Comparing [86] with the corresponding one of the two dimensional approach ([80]) explains why the one dimensional approach is so much simpler then the two dimensional one. Having found the solution for "a" one can now concentrate on the optimal positions for w_1 and w_2. In order to find this minimum one has to differentiate [85] with respect to w_1 and w_2 and setting both differentials to zero. It leads to the following two equations:

$$2 \cdot C(w_2) - C\left(\frac{w_1 + w_2}{2}\right) - C(\infty) = 0 \quad and \quad 2 \cdot C(w_1) - C\left(\frac{w_1 + w_2}{2}\right) - C(-\infty) = 0$$

$$with \qquad \frac{\partial C(x)}{\partial x} = c(x) \qquad [\ 87\ ]$$

If f_w denotes the cost for one warehouse, the total cost for one warehouse c_{one} is easily obtained by [82]:

$$c_{one} = f_w + t \cdot \int_{-\infty}^{\infty} dx\, c(x) \cdot |x - w| \qquad [\ 88\ ]$$

And w has to be obtained from [83]. For two warehouses the total cost is obtained by using [85]:

$$c_{two} = 2 \cdot f_w + t \cdot \left(\int_{-\infty}^{\frac{w_1+w_2}{2}} dx\, c(x) \cdot |x - w_1| + \int_{\frac{w_1+w_2}{2}}^{\infty} dx\, c(x) \cdot |x - w_2| \right) \qquad [\ 89\ ]$$

And w_1 and w_2 are obtained by solving [87]. Chaos occurs if $c_{one} = c_{two}$ or

$$\int_{-\infty}^{\infty} dx\, c(x) \cdot |x - w| = \frac{f_w}{t} + \int_{-\infty}^{\frac{w_1+w_2}{2}} dx\, c(x) \cdot |x - w_1| + \int_{\frac{w_1+w_2}{2}}^{\infty} dx\, c(x) \cdot |x - w_2| \qquad [\ 90\ ]$$

At that point the warehouse arrangement changes from one warehouse to two warehouses within an arbitrary small variation of c(x).

In order to see what happens in a real example I will turn back to customer density distribution of figure 42. For one warehouse it is already known that it should be placed at x = 4.41.

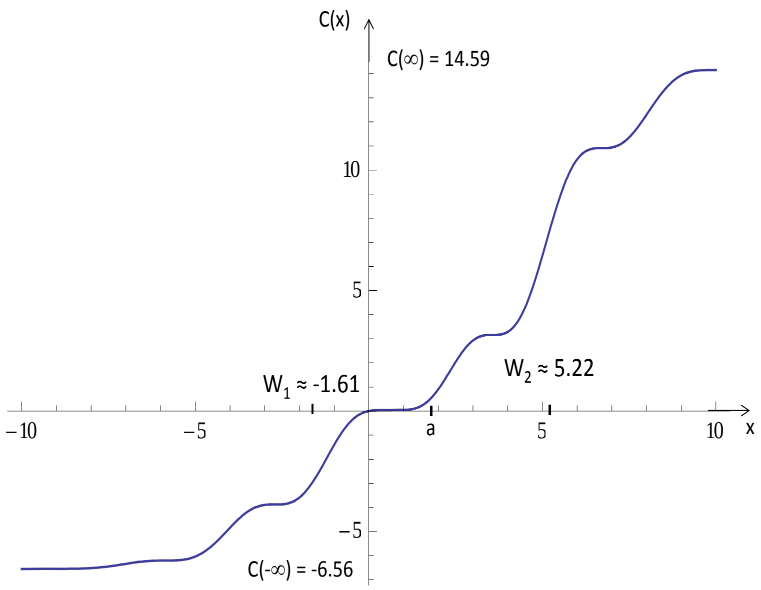

Figure 44: Integral of customer density c(x) in one dimensional case

The total cost c_{one} is then $f_w + 73.6\, c_0\, t$ (cf. figure 43). In order to find the optimal positions w_1 and w_2 one has to find the function C(x) of [87] (it is the integral of c(x)). For the example of figure 42 C(x) is given by figure 44. Obtaining this integral is simple, because a numerical or graphical integration is generally very simple. With it the optimal warehouse positions w_1 and w_2 can be determined by using [87]. Because C(x) is a highly nonlinear function, [87] can be solved numerically or graphically only. Its solution is indicated in figure 44. The first warehouse is placed at x = -1.61 and the second at x = 5.22 leading to a total transport cost of 34.7 $c_0\, t$. Of course, it is much smaller than the total transport cost for one warehouse of 73.6 c_0 t. Chaos occurs if the difference of both transport costs equals the cost for one warehouse. It is graphically shown in figure 45:

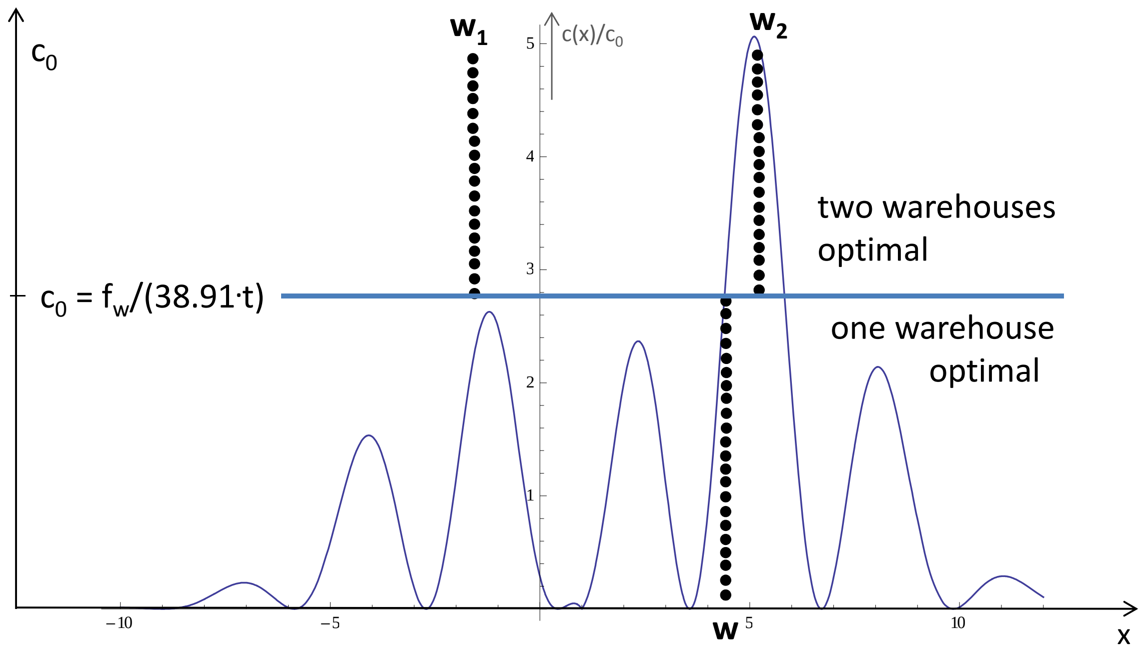

Figure 45: Transition of one to two warehouses (chaos) in 1D customer field

Here the original form of c(x) is kept identical only its prefactor c_0 is change. At a certain c_0 the optimal condition changes from one warehouse to two warehouses as indicated by the blue line in figure 45. Because an arbitrarily small change in c_0 may cause it, the situation of chaos is given.

4. Chaos in learning curves applied to marketing

In section 2 of chapter III, I have discussed learning curves, especially if two sides are learning like in marketing, where at least two sides are fighting for market share. The purpose of this appendix is to state some numbers so that the results can be proven and maybe further analysis can be undertaken.

In figure 21 I have displayed the revenue over 24 month of a new product. Just for the reference the values were:

$\{0,90\}, \{1,86\}, \{2,84\}, \{3,82.5\}, \{4,82\}, \{5,82.5\}, \{6,83\}, \{7,84\}, \{8,85\}, \{9,86\}, \{10,87\}, \{11,88\}, \{12,88.5\}, \{13,89.5\}, \{14,90.5\}, \{15,91.5\}, \{16,92\}, \{17,93\}, \{18,93.5\}, \{19,94\}, \{20,94.5\}, \{21,95\}, \{22,95.5\}, \{23,96\}$

The first coordinate denotes the month (time coordinate) and the second the revenue (in arbitrary units). With this set of $\{t_i, revenue_i\}$ data I have fitted [26]. In a so called least square fit I had to find the minimum of

$$\sum_{i=0}^{23}\left(revenue_i - A - B \cdot e^{-\frac{t_i}{\tau_b}} + C \cdot e^{-\frac{t_i}{\tau_c}}\right)^2 \qquad [\,91\,]$$

by varying the constants A, B, C, τ_b, and τ_c. The canonical way to do it is to differentiate [91] with respect to A, B, C, τ_b, or τ_c, and setting each of the resulting five equations to zero. Having five equations for five unknowns is straightforward to solve. Please note however that these five equations are non-linear (and non-polynomial). So, they show an unknown amount of real and complex solutions. It can be quite tricky to find the one which gives the (absolute) minimum. Because the only possibility is a numerical solution anyway, it is much smarter to find the minimum of [91] directly. Doing so yields the following formula for the revenue development:

$$revenue = 100.2645 + 25.91774 \cdot e^{-\frac{t}{3.062224\,months}} - 36.129483 \cdot e^{-\frac{t}{10.86824\,months}} \qquad [\,92\,]$$

In order to show chaos effects I have produced hundred $\{t_i, revenue_i\}$ sets (each having 24 value pairs) in which the value for the revenue fluctuates Gaussian with a variance of $\sigma^2 = 10^{-6}$. After finding the minimum for one hundred times in [91], one will get one hundred results for the constants A, B, C, τ_b, and τ_c (500 in total). Please note that the numerics can be tricky here, sometimes it does not converge or it converges to the wrong value. Eventually I have found:

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A	B	τ_b	C	τ_c
101.334	22.7792	2.867844012	34.0846	12.36089359
101.297	23.3435	2.959297818	34.6323	12.23271517
101.205	23.1723	2.90076203	34.2905	12.18901258
100.875	22.8913	2.874033965	33.7863	12.02975681
101.001	23.6111	2.904620961	34.5248	11.85800748
100.928	23.7697	2.905583078	34.6486	11.80684939
100.95	24.0188	2.959359121	34.9032	11.78505937
100.821	23.6086	2.85499753	34.3169	11.7490589
100.759	23.549	2.837507307	34.1506	11.73635912
100.815	23.984	2.930523157	34.7587	11.6938959
100.757	23.8909	2.943184761	34.6325	11.64768095
100.833	24.1696	2.949634983	34.8444	11.64052205
100.714	24.0922	2.926612271	34.6774	11.58203672
100.707	24.3996	2.92753759	34.8936	11.53502321
100.625	24.0657	2.940813194	34.6538	11.52983633
100.527	24.0874	2.877052057	34.4138	11.48457679
100.783	24.651	2.933385743	35.1746	11.48014681
100.651	24.3668	2.968222213	34.9737	11.45236617
100.577	24.3963	2.935236932	34.8465	11.42167926
100.576	24.5187	2.987937696	35.0693	11.41967061
100.728	24.7953	3.008396434	35.5155	11.4156033
100.525	24.2763	2.905051303	34.6837	11.41517328
100.555	24.5691	2.989456188	35.1076	11.35523607
100.456	24.6069	2.968275076	34.8949	11.3135866
100.725	25.3775	3.080173845	36.0909	11.31225558
100.632	25.0484	2.981327943	35.5	11.27791142
100.576	25.0244	3.030725495	35.606	11.26288474
100.608	24.9782	2.984041347	35.5207	11.26239004
100.521	24.7846	2.935598833	35.185	11.25710323
100.678	25.4858	3.062027491	36.1901	11.2444326
100.3	24.5135	2.960559427	34.8321	11.1940215
100.529	25.2987	3.032177467	35.7666	11.1829804
100.464	25.0951	2.989849461	35.4339	11.17197354
100.357	24.8273	2.976305631	35.0681	11.16790612
100.462	24.8769	2.980448259	35.309	11.16158628
100.458	25.3576	3.037371823	35.76	11.14237165
100.434	25.2716	3.008224486	35.6112	11.11896852
100.347	25.0844	2.98836629	35.3416	11.11345729
100.359	25.0923	3.069509032	35.6239	11.11104938
100.376	24.9689	2.963964124	35.269	11.10328946
100.492	25.702	3.083412474	36.2146	11.06824013
100.256	25.1641	2.987785931	35.295	11.04757528
100.514	25.5937	3.062055619	36.1152	11.04106171
100.397	25.7754	3.068199948	36.1452	11.00428397
100.493	26.3068	3.104462043	36.7092	10.96766622
100.379	25.9564	3.044232701	36.2084	10.93631238
100.308	25.836	3.107105017	36.2127	10.92826903
100.301	25.7855	3.075304993	36.0382	10.92569001
100.168	25.4624	2.998626629	35.5899	10.88466177
100.32	26.1014	3.066440568	36.2651	10.88393912

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A	B	τ_b	C	τ_c
100.125	25.589	2.986724012	35.5242	10.82744411
100.206	26.1264	3.10099635	36.3259	10.81702037
100.279	26.4329	3.088354741	36.5353	10.81435324
100.238	26.2738	3.115061008	36.4822	10.81193291
100.23	26.4236	3.0884692	36.5757	10.76327785
100.198	26.1859	3.072914106	36.3233	10.75367103
100.13	26.0705	3.101823562	36.2515	10.74981
100.118	25.8775	3.037279569	35.9559	10.74670237
100.248	26.7097	3.142213438	36.8665	10.74334262
100.163	26.3046	3.088841252	36.2687	10.74067709
100.224	26.6676	3.135503938	36.8182	10.72670187
100.239	26.5075	3.107848561	36.747	10.70728009
100.097	26.1941	3.061418171	36.2885	10.70676421
100.214	27.0661	3.132262935	37.1187	10.66556314
100.159	26.9657	3.199426663	37.2809	10.57404368
100.15	27.1926	3.151561756	37.1988	10.57331697
99.9096	26.3549	3.090349457	36.2952	10.54781112
100.142	27.0245	3.131909776	37.113	10.54400752
100.03	26.8007	3.099055098	36.8435	10.51535768
100.1	27.7691	3.225827263	37.8288	10.46370976
100.047	27.5398	3.167243105	37.5437	10.42411557
99.942	27.4494	3.15134326	37.385	10.36411198
100.027	27.7934	3.218072696	37.8697	10.36382197
99.9843	27.7116	3.176236668	37.6499	10.35855088
99.9856	27.6677	3.169391287	37.6396	10.35097035
100.021	28.0246	3.22348762	37.9448	10.34820681
99.8121	27.4308	3.116964077	37.1776	10.27241415
99.9388	27.9797	3.1775891	37.85	10.26317869
99.9469	28.2631	3.219232986	38.1792	10.25956705
99.7311	27.1229	3.085667384	36.7291	10.24950367
99.8816	27.9069	3.203588019	37.7818	10.24581766
99.7728	27.5027	3.135789074	37.2151	10.24324637
99.7767	27.7981	3.129635773	37.3691	10.24045615
99.8536	27.9788	3.191961364	37.835	10.22378851
99.9414	28.6206	3.228462122	38.4718	10.18464766
99.9441	28.7616	3.253714929	38.6183	10.17392322
99.7446	28.7961	3.223726628	38.4328	10.04897872
99.7653	29.3224	3.286997624	39.0803	9.963037132
99.47	28.5205	3.197063817	37.946	9.89070768
99.6566	30.1872	3.349376848	39.8708	9.800462582
99.5118	29.9648	3.270378546	39.3808	9.717795227
99.624	30.5248	3.346720214	40.1496	9.704215511
99.443	29.8721	3.276099869	39.2622	9.70412134
99.4728	30.0193	3.268219507	39.3627	9.697533917
99.4198	29.7946	3.28069997	39.2766	9.674735396
99.5606	30.805	3.357248131	40.315	9.650832867
99.5034	30.5366	3.340046827	40.0746	9.618344106
99.3935	30.2422	3.296924299	39.5614	9.615014807
99.3142	31.5437	3.397870215	40.8952	9.403091737
99.0788	34.0369	3.480742791	43.1836	8.966358224

Appendix

These are the data used for the analysis and lead to the plot of the distribution in figure 23. A statistical analysis of the data has been summarized in the following table:

	$revenue_i$	**A**	**B**	**τ_b**	**C**	**τ_c**
average	na	100.25217	26.346368	3.08311958	36.540725	10.8272137
σ	0.001	0.00449878	0.08124626	0.04371311	0.04777995	0.06287702

As discussed, the standard deviation σ of the conserved quantity "A" (eventual revenue) is roughly four times bigger than the original distribution of revenue. All other quantities have to do with the timely development of the revenue and are not conserved. Not surprisingly their σ values are 40 to 80 times bigger than the variation in input.

5. Solution of diffusion model in marketing

This appendix refers to section 3 of chapter III where chaos effects in the diffusion model of marketing were reported. There are three types of diffusion models in marketing. They are the *exponential diffusion model*, the *logistic diffusion model*, and the *semi logistic diffusion model*. They are represented by [27], [28], and [29]. The goal of these diffusion models is to predict the development of the number of customers, or units sold, or market share, etc. for a newly introduced product. It is described by a variable N(t) or N_t. There are various parameters (a, b, and M) to describe it. The above mentioned equations defined the diffusion models. They are differential equations. In this appendix I will write down these differential equations in their ordinary form and solve them.

In the exponential diffusion model ([27]) $N_t - N_{t-1}$ = dN and dt = 1. Therefore [27] yields:

$$\frac{dN}{dt} = a \cdot M - (a+1) \cdot N \qquad [\ 93\ ]$$

This differential equation is easily solved by separation of variables and integration:

$$\int_{N_0}^{N} \frac{d\tilde{N}}{a \cdot M - (a+1) \cdot \tilde{N}} = \int_{0}^{t} d\tilde{t} \qquad [\ 94\ ]$$

Here N(0) = N_0. Solving the integral yields:

$$t = \frac{ln(a \cdot M - (a+1) \cdot N_0) - ln(a \cdot M - (a+1) \cdot N)}{a+1} \qquad [\ 95\ ]$$

This can be easily solved for N:

$$N(t) = \frac{a \cdot M + (aN_0 + N_0 - a \cdot M) \cdot e^{-t(a+1)}}{a+1} \qquad [\ 96\ ]$$

This is the most general solution of the exponential diffusion model in marketing. Of course, for M = 1 it is identical with [30]. It is now straightforward to solve the other diffusion models. For the logistic diffusion model one gets from [28] the following differential equation:

$$\frac{dN}{dt} = (b \cdot M - 1) \cdot N - b \cdot N^2 \qquad [\ 97\ ]$$

This differential equation is easily solved by separation of variables and integration:

$$\int_{N_0}^{N} \frac{d\tilde{N}}{(b \cdot M - 1) \cdot \tilde{N} - b \cdot \tilde{N}^2} = \int_{0}^{t} d\tilde{t} \qquad [98]$$

Solving the integral yields:

$$t = \frac{ln(1 - b \cdot M + b \cdot N_0) - ln(1 - b \cdot M + b \cdot N) + lnN - lnN_0}{b \cdot M - 1} \qquad [99]$$

Solving for N yields:

$$N(t) = N_0 \cdot \frac{(b \cdot M - 1) \cdot e^{bM \cdot t}}{b \cdot N_0 \cdot e^{bM \cdot t} + (b \cdot M - b \cdot N_0 - 1) \cdot e^{t}} \qquad [100]$$

For M = 1 one gets [31]. And now I come to the most advanced model, the semi logistic diffusion model. It is considered the standard diffusion model in marketing. Starting with [29] gives the differential equation:

$$N \frac{dN}{dt} = a \cdot M + (b \cdot M - a - 1) \cdot N - b \cdot N^2 \qquad [101]$$

Separation of variables and integration yields:

$$\int_{N_0}^{N} \frac{d\tilde{N}}{a \cdot M + (b \cdot M - a - 1) \cdot \tilde{N} - b \cdot \tilde{N}^2} = \int_{0}^{t} d\tilde{t} \qquad [102]$$

Performing the integration yields:

$$t = 2 \frac{\operatorname{artanh} \frac{a - bM + 2bN + 1}{\sqrt{a^2 + 2a(bM + 1) + (bM - 1)^2}} - \operatorname{artanh} \frac{a - bM + 2bN_0 + 1}{\sqrt{a^2 + 2a(bM + 1) + (bM - 1)^2}}}{\sqrt{a^2 + 2a(bM + 1) + (bM - 1)^2}} \qquad [103]$$

Here artanh denotes the inverse function of the hyperbolic tangents tanh $x=(e^x - e^{-x})/(e^x+e^{-x})$. Solving this equation for N yields:

$$N(t) = \frac{\sqrt{a^2 + 2a \cdot (b \cdot M + 1) + (b \cdot M - 1)^2}}{2 \cdot b} \cdot \left[\tanh\left(\frac{t}{2} \cdot \sqrt{a^2 + 2a(b \cdot M + 1) + (b \cdot M - 1)^2} + \operatorname{artanh} \frac{a - b \cdot M + 2b \cdot N_0 + 1}{\sqrt{a^2 + 2a(b \cdot M + 1) + (b \cdot M - 1)^2}}\right)\right] + \frac{b \cdot M - a - 1}{2b} \quad [\ 104\ ]$$

Though this equation looks slightly clumsy, it behaves pretty ordinary. In order to see it I have plotted [104] for M = 100, a = 0.01, and b = 0.02 and various starting values N_0:

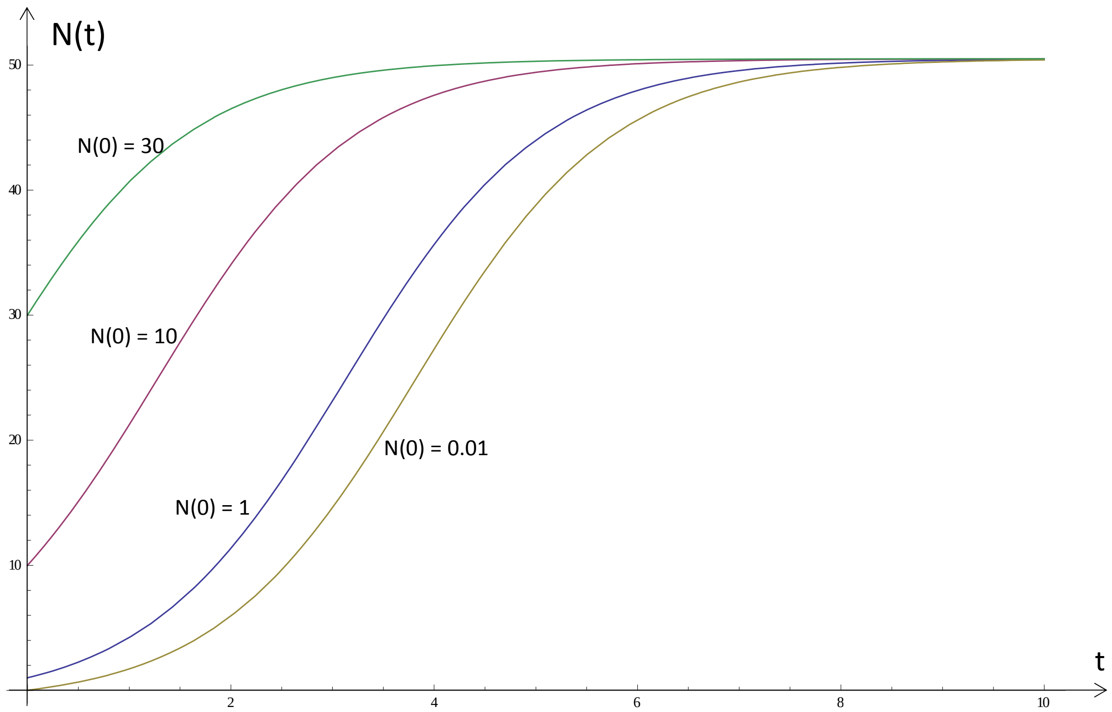

Figure 46: N(t) in semi logistic diffusion model (M = 100, a = 0.01, b = 0.02)

It looks pretty ordinary. Of course, one may choose any other values for the constants and starting values N(0) or N_0. In marketing it is quite often asked for the gain in customers. This is nothing but the timely derivative of [104]. Just for completeness I have calculated it:

$$\frac{\partial N}{\partial t} = \frac{a^2 + 2a(b \cdot M + 1) + (b \cdot M - 1)^2}{4 \cdot b} \cdot sech^2\left[\frac{t}{2} \cdot \sqrt{a^2 + 2a(b \cdot M + 1) + (b \cdot M - 1)^2} + artanh\frac{a - b \cdot M + 2b \cdot N_0 + 1}{\sqrt{a^2 + 2a(b \cdot M + 1) + (b \cdot M - 1)^2}}\right] \quad [\,105\,]$$

Here "sech" is the hyperbolic secant function with sech x = $2/(e^x - e^{-x})$. $\partial N/\partial t$ can be plotted for the same values as in figure 46:

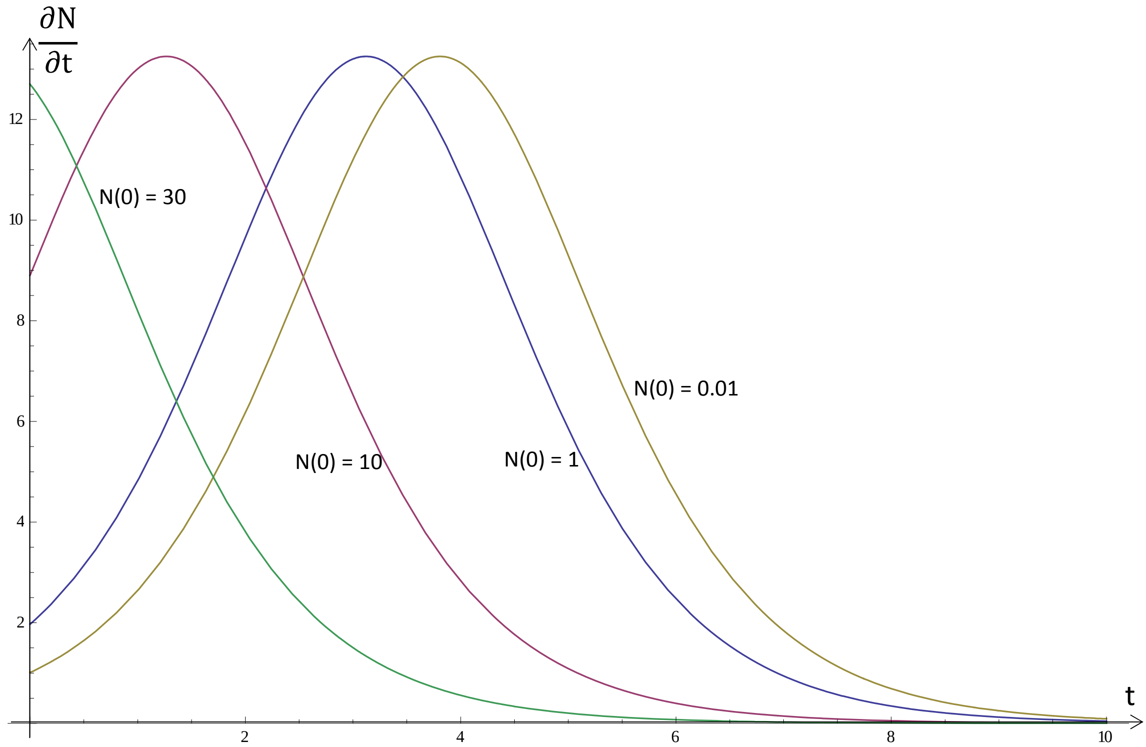

Figure 47: Gain in adaption in semi logistic diffusion model (M = 100, a = 0.01, b = 0.02)

These are the exact solutions of the semi logistic diffusion model. By iterating the map of [29] one gets an approximate result which is sometimes good (cf. figure 24 and figure 25) and sometimes completely unrealistic (cf. figure 26 and figure 27). But even if the iteration gives fair results, the numerical calculations can be tricky. Standard calculation programs like Excel of Microsoft are generally far too inaccurate to give true results. As a rule of thumb, for n iterations one needs to keep 2^n digits. For n = 10 that are 1024 digits (Excel has around 16), for n = 100 that are 10^{30} digits. Even today an ordinary laptop is not powerful enough to do it.

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